Sponsored since 2011 by the
Italian Society for Analytic Philosophy

ISSN 2037-4445
http://www.rifanalitica.it

Rivista Italiana di Filosofia Analitica Junior 9:2 (2018)

Kripke's Modal Logic: A Historical Study

Melissa Antonelli

Abstract. In a very short time Saul Kripke provided a suitable and rigorous semantics for different axiomatic modal systems and established a series of related results. Many key ideas were already in the air in the late Fifties, but it was Kripkean articles' merit to systematically introduce comprehensive devices and solutions. Later on, the spreading of possible-worlds semantics massively changed the approach to modal logic, which enormously increased in popularity after that. Since Kripke's work in modal logic is central to the development of the discipline, the aim of this essay is to present the fundamental results published between 1959 and 1965. Indeed, it was in such a brief and early phase of his career that Kripke was able to conceive the main novelties that would become central to the subsequent academic debates about modality. Here, their presentation will follow the original historical progressive introduction. Particular attention will be given to the interconnection between articles, their similarities in structure and the unified analysis produced by means of them. It actually appears quite impressive that, already in 1959, Kripke seemed to have planned all the developments he would present, one after the other, in the following years. First, an overview of the background where Kripke's ideas start to rise is given. Then, each text's results are individually briefly analysed.

Keywords. Kripke, Modal Logic, Possible-Worlds Semantics, Completeness Proofs.

I wish to thank Sara Negri, Eugenio Orlandelli and the anonymous referees for their valuable comments and suggestions.

Author. Melissa Antonelli, melissa.antonelli@studio.unibo.it.
Received. 11 August 2018. **Accepted**. 19 December 2018.

Introduction

The introduction of possible-worlds semantics completely revolutionised the study of modal logic after their first appearances from 1958–1959 on. Indeed, these, so-called, «marvellous years for possible world semantics» (Copeland 2002, p. 131) paved the way to an increasing interest in the subject. Although at that time some semantics for quantified modal logics and correspondent completeness proofs were being presented by other logicians,[1] it was Kripke who offered a unified tool to analyse different modal systems and to systematically obtain connected results. The enormous popularity of his publications stimulated remarkable steps forwards in many related fields, both under a formal and a philosophical viewpoint.[2]

Given the well-recognized role of Kripke's articles as a turning point in the development of modern modal logic, the aim of the essay is to provide an organic historical reconstruction of the main results published between 1959 and 1965, by particularly focusing on the original sources and by emphasizing the elements of novelty.[3] An overall coherence and connection among his publica-

[1]Particularly remarkable are 1959 Bayart's Henkin-style completeness proof for quantified **S5** and, even more, Hintikka's possible-worlds semantics and completeness proofs for quantified **T** (Kripke's **M**), **S4** and **S5**, presented in a series of seminars held in the Boston area in 1958–1959 (unluckily, the notes of the talks are currently lost). Actually, «it is not clear which of the two [Hintikka and Kripke] was in fact the first to produce a fully worked out completeness proof (it must have been a matter of a few months at most)» (Copeland 2002, p. 130).

[2]Considering, for example, the analysis of counterfactual conditionals, it seems that Prior (1968) and Lewis (1973) proposals have hardly been conceived without the reception of Kripke's semantics (Stalnaker explicitly refers to Kripke's work (Prior 1968, 103, fn. 6), while Lewis affirms to be inspired by the success of possible-worlds semantics, thanks to which «[i]n the last dozen years or so, our understanding of modality has been much improved» (Lewis 1973, p. 418)). Indeed, valuable observations about counterfactual statements have already been advanced by other authors (as Goodman) but it was only by moving the analysis to possible-worlds context that many problems could have been solved. On the other hand, the connection between some foundational problems in philosophy of language or in metaphysics and modal logic is evident.

[3]The title of my essay is too ambitious in two ways. Indeed, in presenting (now standard) Kripke's theories, not all the facets of his modal logic are actually taken into account, due to the vastness of the topic. First of all, my analysis is confined to Kripke's early contributions about formal aspects and, consequently, it is focused on a restricted period of time. Of course, Kripke's philosophy is much wider and other fields are inseparably related (see at least (Kripke 1972) and reflections concerning "paradoxes of identity statements" and names of non-existing entities (Kripke 1963b) and (Kripke 1971)). Although Seventies studies about metaphysics and philosophy of language seem to have born and grown in strict relation to the formal ones, because of their complexity, I will avoid to treat them, confining myself to technical logical aspects. This cut can be partially justified by Kripke's choice – between 1959 and 1965 – to omit, as much as possible, explicit philosophical investigations, see (Kripke 1959a, p. 2) (however, some interests in philosophical questions emerges in (Barcan Marcus 1963b) and (Kripke 1963b)). Secondly, because philosophical issues cannot be possibly split from the «labyrinth full of twist and problems» (Fitting 1999, p. 105) of quantified modal logic, which is the central topic of (Kripke 1963b), I have decided to mainly consider propositional results and to simply outline the content of (*ibid.*) (while treating more in detail (Kripke 1959a)). It must be stressed that some other - extremely relevant but very vast - topics have been completely

tions has interestingly emerged. While many authors have introduced important innovations and at the end of the Fifties other logicians are getting closer to conceiving a relational semantics,[4] what is striking in Kripke's work is the unitary and comprehensiveness of the analysis that, starting from a specific system (quantified **S5**), is extended (with the necessary adjustments) to many other logics. Already in (Kripke 1959b), Kripke consciously planned to deal with all the topics he would debate in the following six years. In a certain way, his articles are rather chapters or parts of a unique patchwork – aimed at systematically clarifying modal logic – and not individually-conceived products. Indeed, although almost each paper can be read independently from the others, the author himself often emphasizes the link with previous and future works.

1 Overview

Formal modal logic enormously expanded during the Twentieth century. Modern interest in it was mainly revived by MacColl's series of articles published in *Mind* from 1880 on and, even more, by C.I. Lewis' publications as starting from 1918. Both authors express their dissatisfaction towards the common notion of material implication.[5] While MacColl does not propose formal definitions or axiomatizations, Lewis begins to introduce various systems of strict implication. In particular, in Appendix II of (Lewis and Langford 1932), he presents the five axiomatic systems **S1-S5**, which rapidly become canonical in the subsequent studies in modal logic.[6] Lewis' original axiomatization does not separate, re-

omitted; in particular, the relationship between Kripke's and category-theoretic semantics has not been treated at all.

[4]In 1946, Carnap proposed a possible-world semantics for quantified **S5**, based on the idea of state-description. In 1947 McKinsey and Tarski gave algebraic characterization for **S4** and **S5**, while the work of Jonsson and Tarski is described by Kripke as the (unaware, see (Copeland 2002, p. 105)) «most surprising anticipation» (Kripke 1963a) of his own (the so-called, "algebraic tradition"). The importance of Kanger's contribute is debated, see (Copeland 2002, pp. 122-123). Hintikka also stated, without proving, soundness and completeness for **T**, **B**, **S4** and **S5** (Kripke quotes Hintikka's research in (Kripke 1959b, p. 324); (Kripke 1963a, 69, fn. 2) and (Kripke 1963b, 83, fn. 1); Hintikka maybe gave some completeness results during Boston seminars). In 1955 Smiley established completeness for **M**. In 1959 Bayart published in French, a Henkin-style completeness proof for **S5**.

[5]In *Symbolical Reasoning* (1897), MacColl claims that $P \supset Q$ and $\neg P \lor Q$ are not equivalent and he distinguishes between extensional and intensional readings of the connectives. However, his texts were not particularly popular (probably because of Russell's wrong interpretation and consequent critique). Anyway, he influences C.I. Lewis, who, in 1912, criticizes Russell and Whitehead's notion of material implication. The so-called "paradoxes" should show its inadequacy to represent the actual and ordinary meaning of implication. Thus, in a series of subsequent articles, Lewis proposes different axiomatic systems of strict implication.

[6]More precisely, Lewis alone wrote Appendix II (but he acknowledges his debts to other authors in (Lewis and Langford 1932, 492, fn. 1)). The systems are numbered in order of strength and weaker systems are contained in stronger ones: **S1** consists of axioms B1-B7, **S2** adds B8 to **S1** (both have been already presented in Ch. 6), **S3** corresponds to A1-A8 (also (Lewis 1918) system), **S4** contains B1-B7 and C10, **S5** contains B1-B7 and C11 (*ivi*, pp. 500-501).

spectively, propositional and modal axioms and rules. This improved presentation – then standard – was first introduced for **S4** by Gödel in the short note in 1933, (Gödel 1933).[7] Differently from (Lewis and Langford, 1932), Gödel opts for necessity (B or N), rather than possibility, as primitive operator.[8] Later on, different presentations and formal systems widespread, such as Fays **T** (1937), equivalent to von Wright **M** (1951), or Lemmon's normal and non-normal systems. It is unlikely that without this "syntactic tradition" Kripke's works would have been conceived in the way they were. Furthermore, such a variety of systems lead a search for more rigorous interpretations of modal notions.

2 A Completeness Theorem in Modal Logic (1959)

According to (Copeland 2002, p. 129), Kripke first becomes interested in modal logics in 1956, after reading (Prior 1956). However, Kripke's relational semantics was not introduced all at once. The main goal of (Kripke 1959a), submitted in 1958 and published in 1959, is to present semantics and completeness theorem for first-order **S5** with equality. Most important novelties introduced are (1) the notion of model as based on a domain **D** and constituted by a set **K** (conceivable worlds) and by the actual world **G** and (2) the definition of the necessary proposition as true in all possible worlds. The notion of validity is defined as disconnected from the one of necessity. Although this apparatus is the basis for subsequent semantics for systems weaker than **S5**, many elements are still missing. Indeed, in 1959 neither accessibility relation nor separate valuation function appears. Moreover, in (Kripke 1963b) some elements are modified.[9] $\mathbf{S5}^{*=}$ completeness proof is given by Beth's semantical tableaux method. This fruitful application of (Beth 1955) technique will be repeated (with some modifications)

[7] In *Eine Interpretation des intuitionistischen Aussagenkalkülus* (presented in 1932 at the Vienna Mathematical Colloquium and published in 1933), Gödel adds a provability operator B ("beweisbar") to a propositional language in order to obtain an interpretation of Heyting's intuitionistic calculus as a logic of provability. He also observes that the given system *G* (propositional system plus axioms T, K, and 4 and rule of necessitation) comes out to be equivalent to Lewis **S4**. The importance of (Gödel 1933) for future (Kripkean and not) developments in modal logic is dual: it introduces the fruitful practice of axiomatizing systems by separating propositional and modal parts and it connects intuitionistic and modal logics. Actually, the idea of introducing a provability predicate was probably suggested him by von Neumann in 1932 (this result was presented in Jan von Plato's "Gödel detective" course, University of Helsinki, 2017).

[8] Goldblatt (Goldblatt 2006, p. 6) notices that, before introducing the diamond operator in (Lewis and Langford 1932), in (Lewis 1918, p. 292) Lewis employs the impossibility operator (~) to define strict implication. The box is devised by Fitch and it first appears in a 1946 paper of Barcan.

[9] In (Kripke 1959a) the model is defined in a domain **D**, while in (Kripke 1963b) the assignment function ψ can assign different domains to different worlds of the same model. For Ballarin (Ballarin 2005), this change is related to a technical problem of 1959 semantics. Indeed, although Prior seemed to have proved the Barcan formula and its converse for quantified **S5**, Kripke suggests that they are not actually derivable (Kripke 1963b), but in (Kripke 1959a, p. 9) he has employed Prior's alleged results (see also (*ivi*, p. 10)).

in the majority of the following articles. Despite the enormous success of this result, not all the passages of the proof are explicit and it sometimes lacks of rigour.

As already said, the propositional system was first introduced in (Lewis and Langford 1932, p. 501). In 1959, Kripke adopts a quantified axiomatization for first order predicate calculus with equality, taken from Rosser, supplemented with modal axioms and rules of inference obtained from (Prior 1956):[10]

A1: $\Box A \supset A$

A2: $\neg\Box A \supset \Box\neg\Box A$

A3: $\Box(A \supset B) \supset (\Box A \supset \Box B)$

R1: If $\vdash A$ and $\vdash A \supset B$, then $\vdash B$

R2: If $\vdash A$, then $\vdash \Box A$

Given a non-empty domain **D** and a formula A, a *complete assignments* for A is defined as a function which assigns an element of **D** to every free individual variable of A, either truth (T) or falsity (F) to every propositional variable of A and a set of ordered n-tuples of members of **D** to every n-adic predicate variable. A *model* of A in **D** is an ordered pair (**G**, **K**), where **K** is a set of complete assignments for A in **D**, **G** ∈ **K**, and every member of **K** agrees with **G** on the assignment of free individual variables of A. The intuitive meaning of this definition is presented only later, in (Kripke 1959a, pp. 2-3). Being **H** a member of **K**, the evaluation of **H** for a compound formula B is inductively defined in the usual way, apart from the modal case: «□B is assigned T if every member of **K** assigns T to B» (*ibid.* p. 2).

[10]Prior takes as the starting point Lewis **S5** formulated by means of Gödel's separation between propositional and modal parts: «It has been shown by Gödel that a system equivalent to **S5** may be obtained if we add to any complete basis for the classical propositional calculus a pair of symbols for 'Necessarily' and 'Possibly', which here will be 'L' and 'M', the axioms G1. CLCpqCLpLq, G2. CLpp, G3. CNLpLNLp; the rule RL: If α is a thesis, so is Lα; and the definition Df. M : M = NLN» (Prior 1956, p. 60). Actually, G1 corresponds to Kripke's (1959) A3 (now-called axiom K) written in Polish notation; G2 to A1 (T) and G3 is A2 (E). However, differently from what Prior seems to argue, this is not the axiomatic system presented in (Gödel 1933). Indeed, the Gödelian 1933 system *G* corresponds to **S4**, not to **S5** (Gödel's 1 ($Bp \supset p$) corresponds to T, 2 ($Bp \supset .B(p \supset q) \supset Bq$) to K, and 3 ($Bp \supset BBp$) to 4). In fact, Prior quote Gödel's text only indirectly, as «cited in R. Feys, *Les systemes formalises des modalities aristotéliciennes* [...] 16.1-16.24» (Prior 1956, 60, fn. 2). In (Feys 1950), Feys presents **S5** subsequently to **S4**: «**16.1** Le système de postulats suivants (Gödel) est un système postulats pour S4 [...] **16.14** :: C Lp LLp. **16.2** Le système obtenu en replaçant le postulat 16.14 par 16.24 ci-dessous est un système de postulats pour S5. **16.24** :: C NLp LNLp» (*ibid*, pp. 499-500). Clearly, Feys' 16.12, 16.13, 16.24 corresponds, respectively, to Prior's G2, G1 and G3. It is likely that the direct source of Prior for **S5** is (Feys 1950) and that he did not consult (Gödel 1933), where **S5** is absent (the first English translation of it seems to be (Hintikka 1969), see (Dawson 1983)). It is possible that, in reading the secondary French source, Prior attributed both systems (introduced one after the other) to Gödel, without guessing that the substitution of **16.14** with **16.24** to obtain **S5** was not another «système de postulats suivants (Gödel)», but an axiomatization provided by Feys himself.

Furthermore, various notions of validity are introduced. A formula A is said to be *valid* in a model (**G**, **K**) of A in **D** iff A is assigned T by **G**.[11] It is *satisfiable* in **D** iff there is some model of A in **D** in which A is valid. It is *valid* in **D** iff A is valid in every model of A in **D**. A is *universally valid* iff A is valid in every non-empty domain. These definitions are required for the completeness proof.

The central concept of the intuitive interpretation is the notion of "possible world" (which is not further analysed). Because in modal logic «we wish to know not only about the real world but about other conceivable worlds» (Kripke 1959a, p. 3), Kripke introduces *models* (**G**, **K**) as composed, not by a single assignment but by a set (**K**) of assignments, one of which (**G**) represents the actual world, and the others embody all the possible ones. Moreover, because, intuitively, a proposition is necessary iff it is true in all conceivable situations, it naturally follows that $\Box$B is defined as true iff B is true in all **K**s. Coherently, a proposition is true in the actual world if it is assigned T by **G**.

After giving possible-worlds semantics for **S5***$^{=}$, Kripke presents his completeness proof. He employs an adaptation to modal logic of the method of semantical tableaux introduced by (Beth 1955). A semantical tableau is «a device for testing whether or not a given formula is semantically entailed by other given formulas» (Kripke 1959a, p. 3) and a formula B is *semantically entailed* by a set of formulas $A_1, \ldots, A_n$ iff A_1 & ...& $A_n \supset$ B is universally valid. Consequently, A_1, ..., A_n do not entail B in case there is a model in which $A_1, \ldots, A_n$ are true and B is not. This situation is represented by a tableau with A_1 & ...& A_n in the left column and B in the right one. Then, the «test of semantical entailment» (Negri 2009, p. 239) proceeds as a systematic search for a countermodel, through the construction of a system of alternative sets of tableaux: the *main tableau* and the *auxiliary tableaux*. The construction is produced by applying usual Beth's rules supplemented by two modal rules, called Yl and Yr.[12] A tableau is *closed* iff either a formula occurs in both its columns or, for some a, $a = a$ occurs in the right column. A set of tableaux is closed iff one of its tableaux is closed.

Then, Kripke establishes a series of proofs in order to obtain, only at the end, completeness for **S5***$^{=}$ (Theorem 7): $\vdash$A in **S5***$^{=}$ iff A is universally valid. This result is obtained by summing Theorem 5 (completeness: if A is universally valid, then $\vdash$A in **S5***$^{=}$) and Theorem 6 (validity: if $\vdash$A in **S5***$^{=}$, then A is universally valid). Both of them require previous proof of Theorem 1, which states that B is seman-

[11] The terminological choice is not particularly satisfying, indeed, in (Kripke 1963a, p. 69, Kripke substitutes the analogous definition for this propositional notion of "*validity* in a model", with "*truth* in a model". This second choice, which for Kripke is «clearly an improvement» (*ivi*, p. 70) and that I will adopt from now on, is the one usually preferred in literature.

[12] See, (Kripke 1959a, p. 4). Kripkean rules are substantially the same as Beth's ones but Kripke presents two rules for each connective (left and right, with rule $\wedge$r the tableau splits and there is nor rule for Ir) and considers only negation, conjunction, universal quantifier and identity, whereas Beth enumerates ten rules and more connectives are employed (Beth 1955, pp. 20-21).

tically entailed by $A_1, \ldots, A_n$ iff the construction beginning with A_1 & ...& A_n in the left column and B in the right one is closed. It is proved by means of lemma 1 (if a construction beginning with A_1 & ...& A_n on the left and B on the right is closed, then B is semantically entailed by $A_1, \ldots, A_n$)[13] and lemma 2 (contraposedly to the "only if" part, if the construction beginning with A_1 & ...& A_n on the left and B on the right is not closed, then B is not semantically entailed by $A_1, \ldots, A_n$, so a countermodel can be found).[14] After presenting Theorems 2 and 3, i.e. the «modal analogues of the Löwenheim-Skolem Theorem» **[22, pp. 6-7]**, and some other results, Kripke defines the *characteristic formula* of a particular stage of a given tableau as A_1 & ...& A_m & $\neg B_1$ & ...$\neg B_n$ (where $A_1, \ldots,$ & A_n are the formulas on the left and $B_1, \ldots,$ & B_n the ones on the right side at the specific stage of the considered tableau). The subsequent Lemma 4 states that if A is the characteristic formula of the initial stage of a construction, and B the characteristic formula of any stage, then $\vdash A \supset B$ in **S5*=**. Finally, Kripke establishes completeness (Theorem 5) by Theorem 1 and Lemma 4: If A is universally valid (so, by Theorem 1, the tableau construction beginning with A in the right column is closed), then $\vdash$A in **S5*=**. Subsequently he proofs validity (Theorem 6) again by using Theorem 1.

Despite the success of the article, (Bayart 1966a) highlights a structural problem in Kripke's completeness proof. Indeed, in both Theorem 1 and 7, «at each step of the construction of a system of tableaux, several possibilities generally occur so that different end results can be reached» (*ivi*, p. 277), e.g., a construction starting with bx and $\neg bx$ in the left column and $(x)bx$ in the right, can either give a closed tableau (working on $\neg bx$) or a not closed tableau (working on $(x)bx$). Bayart suggests supplementing the procedure with a rule imposing a specific choice at each step.[15]

After considering the variety of the logical systems (five distinct ones «proposed in (Lewis and Langford 1932) alone» (Kripke 1959a, p. 13)), Kripke concludes his paper announcing his intention to analyse them in a sequel. Thus, even without introducing the definitive apparatus, already in 1958[16] he plans to extend his semantical approach to obtain completeness for other modal systems. Indeed, in the same year, he publishes an abstract, (Kripke 1959b), where

[13] The proof is a *reductio ad absurdum*. Extremely roughly, if the construction is closed and we assume that B is not semantically entailed by $A_1, \ldots, A_n$ (i.e. there exists a non-empty domain **D** and a model (**G**, **K**) such that A_1 & ...& $A_n \supset$ B is not valid in (**G**, **K**)), then, because of the previously defined tableaux rules, a contradiction follows (the same formula is both true and false in the same model or $a = a$ is false). For further details, see (Negri 2009).

[14] Roughly, for each non-closed construction, it is shown how to define a suitable countermodel, say (**G**, **K**), such that – it is inductively proved – A_1 & ...& A_n are assigned T and B is assigned F (so A_1 & ...& A_n do not semantically entail B). See (Negri 2009). However, in (Kripke 1963a, p. 77), Kripke partially emends this proof.

[15] An emended proof close to Kripke's original one is presented in (Negri 2009, pp. 257-263).

[16] (Kripke 1959a) was received on the 25^{th} of August 1958 (after Prior's revision), but it was probably submitted in March, see (Goldblatt 2006, p. 35).

he explicitly announces all the results of (at least) the subsequent six years. In it he anticipates not only (Kripke 1963a) – where propositional systems weaker than **S5** are modal-theoretically analysed and their completeness is established – but also his following studies of non-normal systems (Kripke 1965b), quantified extensions (Kripke 1963b), and intuitionistic semantics (Kripke 1965a). Interestingly, Kripke quotes Hintikka's preceding works about **S4**, **S5** and **M** (while underling the independence of its own results). Kripke also emphasizes the strict connection between formal analyses (in particular of logics with identity) and widespread issues in philosophy of language (point 3) (e.g. the morning star paradox), which he will deepen during the Seventies.[17]

3 The Undecidability of Monadic Modal Quantification Theory (1962)

The aim of (Kripke 1962) is to show the undecidability of a monadic fragment F of MQ, subsystem of **S5***. This result is defined by Kripke as «*prima facie* surprising», considering the well-known decidability of the monadic predicate calculus for first order logic.

First of all, Kripke presents MQ.[18] Acquaintance with (Kripke 1959a) is explicitly presupposed. Then, the proof of the undecidability of F – fragment of MQ consisting of monadic formulae in two predicate letters – is sketched. Kripke shows how to reduce the decision problem for F to the one of first-order dyadic predicate logic, which is known to be undecidabile. To do so, each closed formula A of extensional dyadic predicate logic is associated to a closed formula A* of F, such that A is valid iff A* is provable in F. Concretely, given any A containing just one dyadic predicate R(x,y), the correspondent modal formula A* in F replaces R(x,y) with $\diamond$(P(x) & Q(y)). It is then established that (1) if A is a valid formula of the dyadic theory, A* is provable in MQ and, conversely, (2) if A is not valid, A* is not provable. Given that decision problem for extensional formulae is reduced to the one of the monadic fragment of MQ, Kripke can concludes that «[s]ince the former decision problem is unsolvable, so is the latter» (*ivi*, p. 115).

After presenting some other modal results, Kripke ends the paper with the general consideration that undecidability cannot be escaped by considering mo dal systems different from **S5***: «it seems unlikely that there will be a *reasonable* modal system in which some formal analogue of this argument could not be

[17] For further details, see (Kripke 1959b, p. 324). Acknowledgment of these topics, central in (Kripke 1971) and (Kripke 1972) is witnessed by Kripke's participation at the Boston Colloquium for the Philosophy of Science 1961/1962 (Barcan Marcus 1963b, 108ss.).

[18] Kripke defines MQ as a system such that: (1) the language is the same as **S5*** (actually, in (Kripke 1962) also $\diamond$ is used); (2) if A contains only &, ¬ and the universal quantifier and is valid, then it is provable in MQ; (3) MQ contains the rule of substitution; (4) MQ is a subsystem of **S5***.

carried out. In the domain of modal logic, decidable monadic systems simply do not arise» (*ivi*, p. 116) (mine italics). In this way, Kripke links formal results to intuitive considerations. Moreover, future extensions of this analysis to intuitionistic predicate calculus are announced.[19]

4 Semantical Analysis of Modal Logic I. Normal Modal Propositional Calculus (1963)

1959 semantics is inadequate for systems weaker than **S5**, so, in order to extend his proposal, two major novelties enter the stages, when, in 1963, Kripke considers propositional systems **T**, **B**, **S4** and **S5**: an accessibility relation between words and an external function ϕ, which assign value to variables relative to worlds (which are no longer complete assignments). Consequently, «the "absolute" notion of possible world in (Kripke 1959a) (where every world was possible relative to every other) gives way to relative notion, of one world being possible relative to another» (Kripke 1963a, p. 70). These innovations will be applied to many other (quantified, intuitionistic, non-normal) logics.[20] Despite the changes, Kripke considers this semantical apparatus as a *generalization* of 1959 one (see, Kripke 1963a, p. 69) and his 1963 work as aimed «to extend the [previous] results» (*ivi*, p. 67). Also the arrangement of the two articles is extremely similar. Furthermore, Kripke announces upcoming treatments of quantificational and non-normal logics (the ones enumerated in (Kripke 1959b)). On ce again, the link with previous and subsequent works is explicit.

The main goal of (Kripke 1963a) is the establishment of completeness for the considered modal propositional systems (still by Beth's method) and of the correspondence between the characteristic axiom of each system and the specific frame property. In defining the *modal propositional calculus* (MPS), in **§1**, Kripke presents the distinction between normal and non-normal systems: a *normal* calculus contains axiom schemes A1 and A3 and rules R1 and R2 are admissible.[21] Starting from the so-defined basic **M**, other normal systems are ob-

[19] Actually, in (Kripke 1965b), Kripke presents a semantics for intuitionistic logic, completeness and decision procedure for the propositional case, but undecidability of monadic quantification logic is not established. It should have been treated in a sequel (Part II) (*ibid.* p. 92) but this has never been written.

[20] It is disputed whether these innovations reflect new philosophical interpretation of modal operators or not. Ballarin interprets these novelties as a technical innovation, so that «[t]here is absolutely no sense in which it is natural to think of such model theoretic constructions (vis-à-vis the 1959 M-models) as better suited to represent a non-semantic notion of metaphysical necessity» (Ballarin 2005, pp. 284-285).

[21] In his review, Kaplan adds to them A0: A, if A is a tautology (Kaplan 1966, p. 120). In (Kripke 1963b, p. 84), Kripke defines normal systems as in 1963, but adds A0: Truth-functional tautologies. Furthermore, in (*ibid.*), he writes that «all systems considered contain all tautologies of classical propositional logic as axiom; thus these axioms will not be listed explicitly» and adds that «[t]his

tained by adding specific axioms (the so-called «reduction axioms» (*ivi*, p. 70)). In a parenthesis, Kripke defines non-normal systems as not satisfying R2. The propositional systems treated in 1963 are **M**, **B**, **S4** and **S5**. It is worth noting that Kripke's basic normal system is **M** (or **T**) of Feys-von Wright (A1-A3 plus R1 and R2)[22] and not **K**, as in recent literature about modal logics.[23] Thus, given that axiom T (A1) corresponds to the frame property of reflexivity, Kripke basic normal modal system comes out as reflexive. System **S4** is obtained by adding the axiom scheme A4 $\vdash \Box A \supset \Box\Box A$ (now 4) to **M**. **B** corresponds to **M** supplemented by the Brouwersche axiom $\vdash A \supset \Box\Diamond A$ (now B).[24] **S5** is defined as in (Kripke 1959a): **M** plus A2: $\vdash \neg\Box A \supset \Box\neg\Box A$. Kripke also notices that **S4** + B is equivalent to **S5**.

In **§2** a *normal model structure*[25] (n.m.s.) is defined as an ordered triple (**G**, **K**, **R**), where **K** is a non-empty set, **G** ∈ **K**, and **R** is a reflexive relation defined on **K**. As already anticipated, the presence of **R** is the novelty that permits Kripke to deal with **M**, **B** and **S4**. Moreover, Kripke explicitly points out the reflexivity of **R** for normal model structure, its transitivity for the **S4**-m.s., its symmetry for Brouwesche-m.s. and its being an equivalence relation for **S5**-m.s. Then, a **M** (**S4**, **S5**, **B**) *model* for a wff A of **M** (**S4**, **S5**, **B**) is obtained by giving a binary function ϕ(P, H) associated with the model structure, from P (atomic subformula of A) and H (subset of **K**) to the set of truth values. The appearance of this evaluation function is the second substantial novelty of 1963 semantics. The inductive definition of ϕ(P, H) is standard for propositional connectives, while for modal formulas it is: $\phi(\Box B, H) = T$ iff $\phi(B, H') = T$ for every H' ∈ K such that H**R**H'. This definition is a coherent extension of 1959 intuitive meaning of necessity. Furthermore, consistently with (Kripke 1959a), a formula A is *true* in a model (**G**, **K**, **R**, ϕ)[26] if it is true in the actual world (i.e. ϕ(A, **G**) = T), otherwise it is false. A is *valid* if it is true in all the models. A is *satisfiable* if true in, at least, one of them.

Similarly to (Kripke 1959a), after the presentation of the formal notions required for completeness proof (presented in **§4**), Kripke gives an informal motivation for its semantics (**§2.1**).[27] Then, he considers the correspondence be-

proviso was inadvertently negated in (Kripke 1963a)».

[22]It was introduced in 1937 as 't' by Feys, who obtained it by dropping 3 in (Gödel 1933), see (Hughes and Cresswell 1966, 50, fn. 7). It was first called **T** in 1953 by Sobocinski, who also showed its equivalence with **M**.

[23]The, now standard, name **K** was given to it by Lemmon and Scott in 1977, in honour of Kripke, and it does not appear in any of Kripke's paper, see (*ibid.*, p. 49, fn. 1). Kripke motivates the reflexivity of his basic system as «an intuitively natural requirement» (Kripke 1963b, p. 84). However, at the end of (Kripke 1963a, p. 95), he considers the possibility to drop it (T) in order to obtain a system for deontic logic.

[24]The formula p ⊃ LMp appears in Becker (1930). The name of axiom (B) derives from Brouwer. For further details, see (Hughes and Cresswell 1966, 70-71, fn. 5).

[25]Kripke never uses the word *frame*. It was employed for the first time in 1968 by Segerberg, apparently under suggestion of Dana Scott. About this, see also (*ibid.* p. 50, fn. 3).

[26]Kripke never uses (G, K, **R**, ϕ) for model, but he writes «in a model ϕ associated with a m.s. (G, K, **R**)» (Kripke 1963a, p. 69).

[27]In this section Kripke himself presents the elements of continuity and discontinuity with (Kripke

tween the characteristic axiom of each system and the related property of **R**. First of all, for each system, «[i]t is clear that every world H is possible relative to itself» (Kripke 1963a, p. 70), so that not only he emphasizes, once again, the reflexivity of basic normal system, but also presents the correspondence between **M** and reflexive structure. Then, he shows (not extremely rigorously and straightforwardly)[28] the ones between, respectively, transitive structure and **S4**, symmetric structure and **B**, and structure where **R** is an equivalence relation and **S5**.[29]

As for (Kripke 1959a), completeness[30] is established by systematic searching for a countermodel. In **§3** Kripke, again, introduces the extended modal variant of Beth's tableaux method. The rules for **M** in (Kripke 1963a) are very close to 1959 ones.[31] As before, the falsification procedure for $A_1 \& \ldots \& A_m \supset B_1 \vee \ldots \vee B_n$ assumes $A_1, \ldots, A_m$ to be true and $B_1, \ldots, B_n$ to be false in the model, by putting $A_1, \ldots, A_m$ to the left and $B_1, \ldots, B_n$ to the right side of the main tableau. Then rules are applied. No order of application is specified (although more rigorous, Kripke considers it as superfluous). For systems **B**, **S4** and **S5**, it is assumed that the relation **R** is (respectively) symmetric, transitive or both. However, Kripke does not treat these rules as formal part of the syntax in tableaux construction (Negri 2009, p. 243).[32] A countermodel for a formula A is then searched by applying the given construction to a main tableau where A is put on the right side. If no countermodel exists, the formula is valid. Still a tableau is *closed* iff a formula appears on both sides (there is no identity in 1963 language); a set of tableaux is closed iff some tableau in it is and a system is closed iff each of its alternative sets is. The continuity between 1959 and 1963 articles is evident. The presentation of the method is followed by some examples of **S4** and **S5**-tabelau construction procedure (**§3.1**).

In **§3.2** Kripke proves the completeness of tableau procedure with respect to

1959a). Kripke defines relational semantics as a generalization of 1959 one, from which it differs for the auxiliary function ϕ and for the appearance of R, whose informal interpretation is given: $H_1\mathbf{R}H_2$ expresses that H_2 is possible relative to H_1 or H_2 is related to H_1. The «"absolute" notion of possible world» for **S5**, is replaced by a «relative notion» (Kripke 1963a, p. 70) (see also (Kripke 1963b, p. 84)). The definition of necessity (and possibility) of A in a world also changes.

[28] The overall strategy to prove correspondence between a frame property and an axiom is to assume the validity of the axiom in the frame, and to show that this frame has the correspondent property. For example, given the Brouwersche axiom B, $A \supset \Box\Diamond A$, and the fact that for an arbitrary H_1, $H_1\mathbf{R}H_2$ holds, Kripke proves $H_2\mathbf{R}H_1$, so that the structure is symmetric.

[29] As an alternative, he proposes to simply abandon **R** and to use the model structure (**G**, **K**).

[30] In (Kripke 1963a, 69, fn. 2), Kripke presents a "status questionis" concerning completeness proof, showing his awareness about previous results and underlying the independence of his discoveries.

[31] Due to the presence of the reflexive relation **R**, the rules $\wedge$r (the one which splits) and the two modal rules change, see respectively (Kripke 1959a, p. 4) and (Kripke 1963a). Of course, propositional 1963 rules do not include $\prod$l, $\prod$r, Il and Ir.

[32] Kripke himself is aware that his presentation lacks in formal clearance. Indeed, in describing $\wedge$r he writes: «I hope this explanation makes the process clear intuitively; the formal statement is rather messy» (*ibid.*, p. 73). See also (Kaplan 1966) critique.

the semantics (for the four propositional systems, a construction for A is closed iff A is valid). This corresponds to 1959 Theorem 1, where semantically entailment substitutes the notion of A-validity.[33] The completeness theorem is developed in **§4**: A is provable in **M**, **B**, **S4** or **S5** iff it is true in the corresponding model. In **§4.1**, validity is proved: every provable formula is valid in the specific system. Kripke defines this constructive proof as an «easy mechanical task», obtained by simply verifying that every axiom is valid for the appropriate theory, and that the rules preserve validity. Then, in **§4.2**, completeness for each system is established.[34] In **§5.1** Kripke proves decidability for **M** and **B** and for **S4** and **S5** in, respectively, analogous ways. In **§6**, Kripke announces future extension of the semantical analysis to non-normal logic and quantification theory.

The legacy of these results is enormous. However, even if Kripke's papers pave the way to a rigorous analysis of modal logic, the most widespread presentation of modal completeness theorems is usually not the original one, based on tableaux method. Indeed, in his 1966 review, Kaplan judges 1959 proof as not clear enough. He alternatively proposes a sketch of a Henkin-style proof, considered «more rigorous» (Kaplan 1966, p. 121) since it avoids tableau technique and does not require any reader's geometrical intuition. Kaplan attributes the idea to Dana Scott. Actually, the first Henkin-style completeness proof for **S5** appeared is 1958 Bayart's paper.[35] Later, completeness proofs for various modal systems appear in Makinson (1966) and in Cresswell (1967). Despite the success of Kaplan's suggestion and the general adoption of Henkin's technique, Negri emphasizes that it hides some information that are instead explicit in Kripke's original proof (Negri 2009). Indeed, elegant Henkin's proof is based on a "trick" and it does not show how to obtain a countermodel for underivable propositions. So, Kripke's proof is more informative. In order to overcome both the lack of data in Henkin's proof and the lack of clearance in Kripke's one, Negri proposes the introduction of a labelled sequent calculus which permits a direct and rigorous completeness proof.[36]

[33] Kripke himself considers 1963 Lemma 1 and 2 corresponds to 1959 homonymous. Lemma 1 establishes validity. As for (Kripke 1959a), the proof is a *reductio ad absurdum*. Lemma 2 demonstrates completeness, by contraposition (and employs König's *Unendlichkeitslemma* is quoted). See, (Negri 2009).

[34] Obtained in **§3.2** the completeness of tableau procedure with respect to the semantics, he has to simply show that «if the construction for A is closed, then A is provable in the appropriate system» (Kripke 1963a, p. 82). He employs Lemma: If A_0 is the characteristic formula of the initial stage of a construction, and B_0 is the characteristic formula of any stage, then $\vdash A_0 \supset B_0$ (analogous to 1959 Lemma 4).

[35] It is striking that Bayart himself, in his review to (Kripke 1959a) does not mention his alternative approach for establishing **S5*** completeness, see (Bayart 1966b).

[36] In (Negri 2009), the labelled calculus is given through internalization of possible-worlds semantics within the syntax. Modal systems stronger than **K** are obtained by adding to G3K rules corresponding to the characteristic properties of the desired systems. This approach, which makes the accessibility relation an explicit part of the syntax and not an implicit property of the tree, simplifies

5 Semantical Considerations on Modal and Intuitionistic Logic (1963)

Kripke completes his formal analysis of normal modal logics in (Kripke 1963b) by considering quantified **M**, **B**, **S4** and **S5**. Despite some changes in model theory, the continuity with previous articles into a unified project is again evident.[37] Moreover, future treatment of non-normal ones is announced. First of all, previous definitions and results for normal **M** and its extensions (definition of model structure and completeness theorem) are summed up. Then, from p. 84 on, Kripke introduces quantifiers by defining a *quantificational model structure* (**G**, **K**, **R**) and a function ψ, which assigns to each H $\in$ **K** a set ψ(H) of all individuals existing in H, called the *domain* of H. ψ(H) needs not to be the same set for different Hs: intuitively, Pegasus does not exist in the real world but may appear in some other. This rises the problem of if and which truth-value to assign to ϕ(P(x), H) when x exists in the domain of some world H' but not in the one of H (Kripkean example is Sherlock Holmes). After comparing different historical proposals of solution to this (Frege-Strawson and Russell), Kripke concludes that «[f]or the purposes of modal logic we hold that different answers to this question represent alternative *conventions*. All are tenable» (*ivi*, pp. 85-86). Kripke opts for the (bivalent) solution «that a statement containing free variables has a truth-value in each world for every assignment to its free variables» (*ibid.*). Differently from previous works, where the content is mainly formal, here the link with general issues in philosophy of language emerges. Kripke also shows that with 1963 semantics the Barcan formula and its converse are not **S5***-valid.

The article concludes with some brief remarks on the "provability" interpretations for propositional modal logics: «Provability interpretations are based on a desire to adjoin a necessity operator to a formal system, say Peano arithmetic, in such a way that, for any formula A of the system, $\Box$A will be interpreted as true iff A is provable in the system» (*ivi*, p. 90). Thus, $\phi(\Box P, F) = T$ iff P is provable in **PA**. Kripke also deals with the mapping of intuitionistic logic into **S4** in order to get a model theory for intuitionistic predicate calculus, without giving its model theory and confining the study to propositional calculus. However, some central elements of (Kripke 1965a) are anticipated: $\neg$A is verified in **E** iff there is no consistent extension of **E** verifying A; A $\supset$ B is verified in **E** iff every consistent extension **E'** of **E** verifying A also verifies B.

Kripke's tableau method. For further details about completeness proof and modal proof theory, see at least (Negri 2005, pp. 312-319), (Negri and von Plato 2001, pp. 81-86) and (Negri 2011).

[37] Apart from explicit and various references to them, the arrangement of the paper is analogous to the former ones (excluded completeness proofs, which are mostly suppressed).

6 Semantical Analysis of Intuitionistic Logic I (1965)

(Kripke 1965b) published in 1965 (but presented in 1963) does not directly concern modal logics but it is still strictly connected with it and to previously mentioned results. Indeed, Kripke writes that «the semantics for modal logic which we announced in (Kripke 1959b) and developed in (Kripke 1963a) and (Kripke 1963b), together with the known mapping of intuitionistic logic into the modal system **S4**, inspired the present semantics for intuitionistic logic» (*ivi*, p. 92).

In **§1** intuitionistic semantics is presented. An intuitionistic model structure is an ordered triple (**G**, **K**, **R**), with **R** reflexive and transitive. Kripke adds to the usual definition a condition to be satisfied: if ϕ(P, H) = T and H**R**H', then ϕ(P, H') = T (H, H' ∈ **K**). The truth-value of a formula in a world (H) is defined in the standard way for & and ∨, while for negation and implication, it is: ϕ(A ⊃ B, H) = T iff for all H' ⊃ **K** such that H**R**H', ϕ(A, H') = F or ϕ(B, H') = T; ϕ(¬A, H) = T iff for all H' ∈ **K** such that H**R**H', ϕ(A, H') = F. Then, the usual notion of validity and quantificational model are presented. Kripke explicates that **G** has to be interpreted as the "evidential situation". Given H to be any situation, we have H**R**H' if, as far as we know at the time H, we may later get enough information to advance to H'. Thus, «[t]he requirement that, for any A, if ϕ(A, H) = T and HRH', then ϕ(A, H') = T simply means that if we already have e proof of A in the situation H, then we can accept A as proved in any later situation H'» (Kripke 1965b, p. 99). The interpretation of the connectives in intuitionistic semantics is explained as well: «To assert ¬A intuitionistically in the situation H, we need to know at H not only that A has not been verified at H, but that it cannot possibly be verified at any later time, no matter how much more information is gained [...] Again, to assert A ⊃ B in a situation H, we need to know that in any later situation H' where we get a proof of A, we also get a proof of B» (*ibid.*).

After presenting tableaux method for intuitionistic logic (**§2**), validity (**§3.1**) and completeness are established by Beth's method (**§3.2**).[38] Results concerning decidability (decision procedure for propositional intuitionistic logic and undecidability of monadic quantification theory) should have appeared in a sequel, Part II, but this has never been published.

[38] The analogy with 1963 proof is evident. The proof of Theorem 2 (the completeness of tableau procedure) is just sketched because it is «a routine variation of the proofs of the corresponding theorems in (Kripke 1963a) [Lemma 1 and 2]». Validity (Theorem 3) is the same «trivial» (Kripke 1965b, p. 214) task, as the analogous «mechanical» (Kripke 1963a, p. 82) proof presented in **§4.1** two years before. Also completeness proof (Theorem 4) is defined as similar to 1963 one.

7 Semantical Analysis of Modal Logic II. Non-normal Modal Propositional Calculi (1965)

Kripke also introduced a special kind of worlds, dubbed *non-normal worlds*, in order to provide a semantics for modal logics weaker than the basic **K** (called "non-normal" in their turn), such as C.I. Lewis **S2** and **S3**. Specifically, the non-normal modal systems at issue do not include the rule of necessitation: if $\vdash$A, then $\vdash \Box$A (Kripke 1963a, p. 67). For this reason, to Kripke they appear to be «intuitively somewhat unnatural» (*ivi*, p. 206). Nevertheless, he considers useful to propose an «elegant model theory» (*ibid.*) for them, where non-normal worlds appears to be technical devices to make necessitation fail (Berto 2013).

Their treatment in (Kripke 1965a) requires acquaintance of (Kripke 1963a) is presupposed.[39] In **§2**, Kripke defines some non-normal propositional systems[40] and, in **§3**, their semantics. He starts showing that in **E2**, **E3** no formula of the form $\Box$B is provable, not even $\Box$(A $\supset$ A) (although A $\supset$ A is true in every world). Moreover, in **E2**, $\vdash \Box$B $\supset \Box$(A $\supset$ A). Consequently $\vdash \neg\Box$(A $\supset$ A) $\supset \neg\Box$B. Thus, if A $\supset$ A is not necessary, nothing is. This leads Kripke to divide possible worlds in two classes: "normal worlds", where necessity is evaluated according to 1963 semantics, and "non-normal worlds", where $\Box$B is always false. Then, (**E2**) *model structure* is defined as a quadruple (**G**, **K**, **R**, **N**), where **K** is the set of worlds, **G** $\in$ **K**, **N** $\subset$ **K**, and **R** is reflexive on **N**. **N** represents the set of normal worlds.[41] A *model* is obtained by associating a valuation function ϕ(P, H) to the frame. ϕ(A, H) is inductively defined as in 1963 for propositional connectives (&, $\neg$), but it changes for modal formulas. Indeed, a modal formula is true in a normal world if it results necessary in it in the usual sense (otherwise it is false), but it is always false in non-normal worlds. Again, a formula A is true in a model (**G**, **K**, **R**, **N**, ϕ) iff ϕ(A, G) = T and A is valid iff it is true in every model. Differently from (Kripke 1963a), the distinguished **G** plays here an essential role in the definitions.[42]

[39] Kripke writes that (Kripke 1965a) «continues the investigations of (Kripke 1963a)» and that it «extends the results of (*ibid.*) to these and other systems. The results of this paper were announced in (*ibid.*), (Kripke 1959b)» (*ibid.*, p. 206). Titles himself emphasize the continuity between these two works.

[40] Lemmon **E2** is characterized by axioms A1 (T) and A3 (K) and rules R1 and (Eb) If $\vdash$A $\supset$ B, then $\vdash \Box$A $\supset \Box$B (Lemmon presents, in 1957, the first "Gödel-style" formulation of *non-normal* **S1**-**S3** and introduces **E1**-**E5** as the "epistemic" counterparts of **S1**-**S5**). **E3** is obtained from **E2** by replacing A3 with stronger (1) $\Box$(A $\supset$ B) $\supset \Box$($\Box$A $\supset \Box$B). **E4** consists of axioms A1, A3 and A4 (4) and rules R1 and (Eb). The re-defined **E5** is constituted by A1, A3 and A2, i.e. $\Box$A $\supset$ ($\neg\Box$B $\supset \Box\neg\Box$B) and rules R1 and (Eb) (original **E5** collapses into **S5**). Łukasiewicz Ł-modal system is axiomatized by A1, (4), i.e. (A $\supset$ B) $\supset$ ($\Box$A $\supset \Box$B), plus R1.

[41] **E3**-m.s. are obtained when **R** is transitive; **S2** (**S3**)-m.s. are **E2** (**E3**) where **G** is normal.

[42] Indeed, for example, $\Box$(A $\supset$ A) is valid in **S2** (where **G** is normal), because ϕ($\Box$(A $\supset$ A), **G**) = T. Then, in **§3** and **§4** Kripke presents tableaux method and semantical rules for non-normal propositional logics, in **§5** completeness theorem and in **§7** (according to (Makinson 1970), somehow inaccurately) other non-normal modal systems widespread in literature.

References

Ballarin, Roberta (2005). "Validity and Necessity". In: *Journal of Philosophical Logic* 34.3, pp. 275–303.

— (2017). *Modern Origins of Modal Logic*. Ed. by Edward N. Zalta. The Stanford Encyclopedia of Philosophy. URL: https://plato.stanford.edu/entries/logic-modal-origins/.

Barcan Marcus, Ruth (1963a). "Discussion". In: *Boston Studies in the Philosophy of Science: Proceedings of the Boston Colloquium for the Philosophy of Science 1961/1962, vol. I*. Ed. by M. Wartofsky. Dordrecht: Reidel, pp. 105–116.

— (1963b). "Modal Logic I: Modalities and Intensional Languages". In: *Boston Studies in the Philosophy of Science: Proceedings of the Boston Colloquium for the Philosophy of Science 1961/1962, vol. I*. Ed. by M. Wartofsky. Dordrecht: Reidel, pp. 77–96.

Bayart, Arnauld (1966a). "Review: Saul A. Kripke, A Completeness Theorem in Modal Logic". In: *The Journal of Symbolic Logic* 31.2, pp. 267–277.

— (1966b). "Review: Saul A. Kripke, The Undecidability of Monadic Modal Quantification Theory". In: *The Journal of Symbolic Logic* 31.2, pp. 277–278.

Berto, Francesco (2013). *Impossible Worlds*. Ed. by Edward N. Zalta. The Stanford Encyclopedia of Philosophy. URL: https://plato.stanford.edu/entries/impossible-worlds/.

Beth, Evert Willem (1955). "Semantic Entailment and Formal Derivability". In: *The Philosophy of Mathematics*. Ed. by J. Hintikka. Oxford 1969: Oxford University Press, pp. 9–41.

Borghini, Andrea et al. (2010). *Il genio compreso: La filosofia di Saul Kripke*. Roma: Carocci.

Copeland, Brian Jack (2002). "The Genesis of Possible Worlds Semantics". In: *Journal of Philosophical Logic* 31.1, pp. 99–137.

Dawson, John William (1983). "The Published Work of Kurt Gödel: An Annotated Bibliography". In: *Notre Dame Journal of Formal Logic* 24.2, pp. 255–284.

Feys, Robert (1950). "Les systemes formalises des modalities aristotéliciennes". In: *Revue philosophique de Louvain* 20.1, pp. 478–509.

Fitting, Malvin (1999). "On Quantified Modal Logic". In: *Fundamenta Informaticae* 39.1, pp. 105–121.

Gabbay, Dov (1969). "Review: Saul A. Kripke, Semantical Considerations for Mo dal Logics". In: *The Journal of Symbolic Logic* 34.3, pp. 501–519.

Ghilardi, Silvio (1991). "Incompleteness results in Kripke semantics". In: *The Journal of Symbolic Logic* 56.2, pp. 517–538.

Goldblatt, Robert (2006). "Mathematical Modal Logic: A View of its Evolution". In: *Handbook of the History of Logic, vol. 7*. Ed. by M. Wartofsky. New York: Elsavier, pp. 1–97.

Gödel, Kurt (1933). "An Interpretation of Intuitionistic Sentential Logic". In: *The Philosophy of Mathematics*. Ed. by J. Hintikka. Oxford 1969: Oxford University Press, pp. 128–129.

— (2003). *Collected Works, Volume V Correspondence H-Z*. Oxford: Clarendon.

Hintikka, Jaakko (1969). *The Philosophy of Mathematics*. Oxford: Oxford University Press.

Hughes, George Edward and Maxwell John Cresswell (1966). *A New Introduction to Modal Logic*. London-New York: Routledge.

Kaplan, David (1966). "Review: Saul A. Kripke, Semantical Analysis of Modal Logic I. Normal Modal Propositional Calculi". In: *The Journal of Symbolic Logic* 31.1, pp. 120–122.

Kontchakov, Roman (2005). "Undecidability of first-order intuitionistic and modal logics with two variables". In: *Bullettin of Symbolic Logic* 11.3, pp. 428–438.

Kripke, Saul (1959a). "A Completeness Theorem in Modal Logic". In: *The Journal of Symbolic Logic* 24.1, pp. 1–14.

— (1959b). "Semantical Analysis of Modal Logic (abstract)". In: *The Journal of Symbolic Logic* 24.1, pp. 323–324.

— (1962). "The Undecidability of Monadic Modal Quantification Theory". In: *Zeitschritft für matheatische Logik und Grundlagen der Mathematik* 8.2, pp. 113–116.

— (1963a). "Semantical Analysis of Modal Logic I. Normal Modal Propositional Calculi". In: *Zeitschritft für matheatische Logik und Grundlagen der Mathematik* 9.5, pp. 67–96.

— (1963b). "Semantical Considerations on Modal Logic". In: *Acta Philosophica Fennica* 16.1, pp. 83–94.

— (1965a). "Semantical Analysis of Intuitionistic Logic I". In: *Formal Systems and Recursive Functions: Proceedings of the Eight Logic Colloquium (Oxford 1963)*. Ed. by J.N. Crossley and M. Dummett. Amsterdam: North Holland, pp. 93–130.

— (1965b). "Semantical Analysis of Modal Logic II. Non-Normal Modal Propositional Calculi". In: *Theory of Models*. Ed. by J. Addinson, L. Henkin, and A. Tarski. Amsterdam: North Holland, pp. 206–220.

Kripke, Saul (1971). "Identity and Necessity". In: *Identity and Individuation*. Ed. by M.K. Munitz. New York: New York University Press, pp. 135–164.

— (1972). *Naming and Necessity*. Cambridge, Mass.: Harvard University Press.

Lewis, C.I. and C.H. Langford (1932). *Symbolic Logic*. New York: Century Co.

Lewis, Clarence Irving (1918). *A Survey of Symbolic Logic*. Berkeley: University of California Press.

Lewis, David (1973). "Counterfactuals and Comparative Possibility". In: *Journal of Philosophical Logic* 2.4, pp. 418–446.

Lindström, Sten and Krister Segerberg (2007). "Modal Logic and Philosophy". In: *Handbook of Modal Logic*. Ed. by P. Blackburn. New York: Elsevier, pp. 1149–1214.

Makinson, David (1970). "Review: Saul A. Kripke, Semantical Analysis of Modal Logic II. Non-Normal Modal Propositional Calculi". In: *The Journal of Symbolic Logic* 35.1, pp. 135–136.

Mugnai, Massimo (2013). *Possibile/necessario*. Bologna: Il Mulino.

Negri, Sara (2005). "Proof analysis in modal logic". In: *Journal of Philosophical Logic* 34.1, pp. 507–544.

— (2009). "Kripke completeness revisited". In: *Acts of Knowledge – History, Philosophy and Logic*. Ed. by G. Primiero and S. Rahman. College Publications, pp. 233–266.

— (2011). "Proof theory for modal logic". In: *Philosophy Compass* 6.8, pp. 523–538.

Negri, Sara and Jan von Plato (2001). *Structural Proof Theory*. Cambridge: Cambridge University Press.

— (2011). *Proof Analysis: A Contribution to Hilbert's Last Problem*. Cambridge: Cambridge University Press.

Prior, Arthur Norman (1956). "Modality and Quantification in S5". In: *The Journal of Symbolic Logic* 21.1, pp. 60–62.

— (1968). "A Theory of Conditionals". In: *American Philosophical Quarterly* 2.1, pp. 98–112.