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A Formal Analysis of the Best System Account of Lawhood

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Abstract. In this work I attempt a reformulation of Lewis' Best System Account, explicitating the underlying formal conception of scientific theories and trying to define the concepts of simplicity, strength and balance. This essay is divided in three sections. In the first one I introduce the Best System Account of natural laws and formulate the need for its improvement. In the second section I outline a formal framework where the notions of deductive system and scientific theory can be defined precisely. In the last section the notions of simplicity, strength and balance are analyzed. To conclude I argue that the framework proposed does indeed provide the precision required. In addition, it also offers interesting insights on the plurality of concepts of simplicity, strength and balance, and on the general enterprise of formalizing scientific theories.

Keywords. Best System Account, Theory Choice, Formalization of Scientific Theories..

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1 The Best System Account

The Best System Account, BSA hereafter, is an attempt to answer the philosophical question: "What are natural laws?". The three philosophers associated with this perspective on natural laws are J.S. Mill, F.P. Ramsey and D. Lewis, and for this reason BSA is also known as MRL account. Let us introduce BSA quoting the *locus classicus* of the latter author. In his 1973 book *Counterfactuals*, Lewis characterized BSA in the following terms:

> Whatever we may or may not ever come to know, there exist (as abstract objects) innumerable true deductive systems: deductively closed, axiomatizable sets of true sentences. Of these true deductive systems, some can be axiomatized more *simply* than others. Also, some of them have more *strength*, or *informational content*, than others. The virtues of simplicity and strength tend to conflict. Simplicity without strength can be had from pure logic, strength without simplicity from (the deductive closure of) an almanac. [...] What we value in a deductive system is a properly balanced combination of simplicity and strength - as much of both as truth and our way of balancing will permit. We can restate Ramsey's 1928 theory of lawhood as follows: a contingent generalization is a law of nature if and only if it appears as a theorem (or axiom) in each of the true deductive systems that achieves a best combination of simplicity and strength. (Lewis 1973, p. 73, original italic)

We can immediately observe that Lewis reduces the problem of characterizing natural laws to the problem of theory choice: once we have selected the best system(s) we can determine if a statement is a natural law by checking if it is a theorem or an axiom of said system(s). It is worthwhile to remark that this procedure will fail if the systems we are considering are undecidable.

Lewis' conception itself was not monolithic. It was articulated and slightly modified during time in order to make it fit in Lewis' own philosophy, e.g. with Principal Principle, modal realism and natural properties.[1] In what follows, however, I won't analyze the development of Lewis ideas through time. My aim is to discuss, and possibly clarify, the four core notions of BSA, namely the notions of deductive system, simplicity, strength and balance. As can be seen from the last quotation, for Lewis simplicity and strength are binary relations such that:

- a system is simpler than another one if it has a simpler axiomatization;
- a system is stronger than another one if it has more informational content.

From other textual evidences it seems that for Lewis, given a deductive system, the addition of an assumption increases the strength and decreases the simplic-

[1] See (Lewis 1973, 1986, 1994, 1999).

ity of the deductive system. I therefore take the *number* of axioms (or hypotheses, as I will prefer to call them later) to be the Lewisian measure of the symplicity of a deductive system.[2]

This characterization is insufficient, as I will argue in what follows. Indeed, the necessity to pin down these concepts more precisely can be traced back to Lewis himself, as witnessed by the following quotations:

> In science we have standards - *vague* ones, to be sure - for assessing the combination of strength and simplicity offered by deductive systems. (Lewis 1973, pp. 73–74, emphasis mine)

and

> Of course, it remains an unsolved and difficult problem to say what simplicity of a formulation is. (See the 1983 article "New work for a theory of universals", reprinted in Lewis 1999, p. 42)[3]

In order to pursue the analysis of these notions I will stick to the 1973 formulation of BSA. This is, to the best of my knowledge, faithful enough to the version of BSA that was received in the literature on natural laws.[4]

1.1 The contemporary debate and the need for a more precise version of BSA

The contemporary literature on BSA addresses a wide range of issues, essentially accepting the 1973 formulation and its core notions. In general, we can identify roughly two attitudes towards the explicit definitions of simplicity, strength and balance. On one hand, the issue is ignored, in the sense that scholars rest content with Lewis' characterization or simply decide to postpone its analysis (among the others, the articles (Cohen and Callender 2009), (Jaeger 2002) and (Robert 1999) are, in different degrees, examples of this perspective). On the other hand, it is perceived as problematic (see for example (Psillos 2002, p. 152); (Bird 1998, p. 40); (Armstrong 1983, p. 67); (Mumford 2004, p. 44)). The clearest exposition of this second stance is Van Fraassen's:

> I have written here as if simplicity, strength and balance are as straightforward as a person's weight or height. Of course they are not, and the literature contains no account of them which it would be fruitful to discuss here. [...] To utilize these motions uncritically, as if they dealt with such well-understood triads as 'under five foot five, over

[2] The correctness of this interpretation is however not essential for the aim of this paper, namely providing an apt framework to specify the notions of simplicity, strength and balance.

[3] Where 'formulation' refers to the formulation of a deductive system.

[4] See for examples, among the recent papers, (Bird 2008, p. 74) and (Cohen and Callender 2009, p. 4).

> 200 pounds, overweight' may be unwarranted. (Van Fraassen 1989, pp. 41–42)

I agree with this concern and I take the insufficient precision of such notions as a drawback of BSA. The following section will be devoted to the (re)construction of a suitable frame for such tasks.

2 The Formal Framework

To attempt a clearer formulation of simplicity, strength and balance we have to use a toolkit of more precise, and possibly shared, definitions. According to Lewis, these notions are to be applied to scientific theories conceived as deductive systems. But what is a deductive system exactly? In his words a deductive system is a "deductively closed, axiomatizable sets of true sentences" (Lewis 1973, p. 73). However, a deductive system is usually understood as a purely syntactic object.[5] What is then the role of truth in a formal representation of scientific theories and what do we mean by deductive system? Given that BSA is essentially a formal account of lawhood, the notions of axiomatization, derivation and deductive system are crucial. But Lewis is not explicit in explaining how they enter the picture. I maintain that we need a more precise formal framework. This is not just a concern about tidiness: we need an improved version of BSA to evaluate BSA itself, its assumptions and its consequences. Questions like

- what conception of scientific theories is required by BSA?
- how do standards of simplicity and strength look like?
- how do we calculate the balance of a deductive system?

cannot be addressed employing the 1973 formulation of BSA. In what follows I will provide an aswer to the first two questions and suggest possible replies to the third one.

To this end in the rest of this section we will attempt a reconstruction of BSA. Assuming that scientific theories can be formalized, we treat them as theories in model-theoretic sense.[6] To add further generality, we abstract from a particular deductive system (in Model Theory it is usually first order classical logic) using a general theory of logical calculi such as the one developed in Abstract Algebraic Logic.[7] This latter step enables us to vary the inferential environment in which a scientific theory lives and study the consequences.

[5] See (Font, Jansana, and Pigozzi 2003, p. 5 and subsection 2.2).

[6] The founding fathers of this approach are, among the others, Tarski and Carnap, see (Tarski 1944, pp. 346–347), (Tarski 1994) and (Carnap 1937). For more recent considerations on this stance see (da Costa and French 2000), for a classic text of Model Theory see (Chang and Keisler 1990).

[7] See (Font, Jansana, and Pigozzi 2003).

2.1 Logical languages and formulas

Prior to outlining the definition of deductive system, let us define a formal language along the lines of Johnstone's presentation.[8] For the sake of simplicity I will stick to first order languages (for a definition of language appropriate for higher order logic see (Johnstone 2002, p. 940)). Each language can have non-logical symbols for basic sorts, functions and relations: these symbols constitute the signature of the language. A *signature* Σ is thus composed of:

1. A set Σ-Sort of *sorts*, symbols for kinds or families of objects.
2. A set Σ-Fun of *function symbols* together with a map assigning to each function symbol its *type*, a finite non empty list of sorts (where the last sort is the sort of the output). We write $f\colon A_1 \dots A_n \to B$ to indicate that f has type $A_1 \dots A_n B$ and call n the *arity* of f. If $n = 0$ f is called a *constant* of sort B.
3. A set Σ-Rel of *relation symbols* together with a map assigning to each relation symbols its type, a list of sorts as in the previous case. We write $R\colon A_1 \dots A_n$ to indicate that R has type $A_1 \dots A_n$ and call n the arity of R. If $n = 0$ R is called an *atomic proposition*.

For each sort A of Σ-Sort we assume to have a countably infinite number of variables of sort A. We now define the *terms* of a language and their sorts recursively (we write $t\colon A$ to indicate that t is a term of sort A):

1. $x\colon A$ if x is a variable of sort A.
2. $f(t_1, \dots, t_n)\colon B$ if $f\colon A_1 \dots A_n \to B$ and $t_1\colon A_1, \dots, t_n\colon A_n$.

Note that for the second clause constants are terms. The terms are those collections of symbols of the language that stand for individuals (even though they do not always denote a specific one).

The next step is to define the formulas of the language, but to do that we first have to introduce the logical symbols. Roughly speaking[9], logical symbols are defined by a set Con of *quantifiers* and *connectives symbols* together with a map assigning to each connective symbol a natural number n corresponding to its arity. A *language* L is thus composed of a signature Σ, a set Con with the relative map and a set of auxiliary symbols (such as brackets). With the aid of logical symbols we can finally define the set of formulas Fm_L of the language L in the usual recursive fashion:

1. $R(t_1, \dots, t_n)$ belongs to Fm_L if R is a relation of type $A_1, \dots, A_n$ and $t_1\colon A_1, \dots, t_n\colon A_n$.

[8] See (Johnstone 2002, p. 808).

[9] For the sake of brevity we avoid a precise discussion of free and bounded variables. This discussion is inessential for our purposes and these notions should be clear to anyone familiar with basic logic. See (Johnstone 2002, p. 809) for details.

2. $c(\phi_1, \ldots, \phi_n)$ belongs to Fm_L if c is an n-ary connective and $\phi_1, \ldots, \phi_n$ are formulas.

3. $qx.\phi(x)$ belongs to Fm_L if q is a quantifier and $\phi(x)$ is a formula with free variable x.

The formulas obtained via the first condition are called *atomic formulas*. By definition they are completely independent from the choice of connectives. The set Fm_L is thus generated combining atomic formulas by means of connectives and quantifiers. In general, formulas are assertions about individuals.

2.2 Deductive systems and theories

Now that we have all the linguistic notions in place, let us turn to the definition of deductive system. Following (Font, Jansana, and Pigozzi 2003), a *deductive system* or a *logic* in a language L is a pair $S = \langle Fm_L, \vdash_S \rangle$ where $\vdash_S$ is a substitution invariant *consequence relation* on Fm_L, i.e., a relation $\vdash_S \subseteq \wp(Fm_L) \times Fm_L$ satisfying:

1. if $\phi \in X$ then $X \vdash_S \phi$.

2. if $X \vdash_S \phi$ for all $\phi \in Y$ and $Y \vdash_S \psi$ then $X \vdash_S \psi$.

Intuitively $\vdash_S$ represents all the inferential procedures of a deductive system. When such relation holds between a set of formulas Γ and a formula ϕ we write $\Gamma \vdash_S \phi$ to mean that we can derive the formula ϕ, the conclusion, applying the inferential procedures of $\vdash_S$ to the formulas in Γ, the premises. In general, a deductive system is nothing more than a machinery to make proofs in a certain language, it is a purely syntactical inferential engine.

As this definition shows, a deductive system is dependent on the language, or, more precisely, on the set of formulas generated by a certain language. But there is, as we have seen, a distinction between logical and non-logical symbols, between the set Con and the signature of a language. The reason for this distinction is that a deductive system is dependent on the connectives and quantifiers but not on the signature. Logical symbols play an essential role in inferential processes, while the non-logical symbols are idle in this respect.

The *theorems* of S are the formulas ϕ such that $\emptyset \vdash_S \phi$, that is, the formulas that can be proved without any premise. There are different ways to present a deductive system: for example as an axiomatic calculus, as a natural deduction calculus or as a sequent calculus. Given that the issue of the number of axioms is important in Lewis' definition of the criterion of simplicity, let us spend a few words on the axiomatization of deductive systems (we will return to the problem in Subsection 3.1.1). A *Hilbert-style calculus* is a pair $P = \langle Ax, Ru \rangle$ consisting of a set of axioms and a set of inference rules, where by 'inference rule'we mean any pair $\langle \Gamma, \phi \rangle$ and by axiom a rule of the form $\langle \emptyset, \phi \rangle$ (which is usually written simply as ϕ). In what follows we will use the term 'inference rule' to refer to inference rules stricto sensu, not to axioms.

A pair $\langle Ax, Ru \rangle$ is a *presentation* of a deductive system S if $\Gamma \vdash_S \phi$ iff ϕ is contained in the smallest set of formulas that includes Γ together with all substitution instances of the axioms of Ax, and is closed under direct derivability by the inference rules in Ru.

The same deductive system can have different presentations: given two presentations P_1 and P_2 in the same language, it is sufficient that the consequence relation $\vdash_1$ associated with P_1 is the same as the consequence relation $\vdash_2$ associated with P_2. This for example happens when, given the same inference rules and two different sets of axioms Ax_1 and Ax_2, we can derive all the axioms of Ax_1 from Ax_2 and vice versa.

We define an S-*theory* (or just a *theory*, when S is understood) as a set of formulas Γ closed under the consequence relation $\vdash_S$, i.e., such that if $\Gamma \vdash_S \phi$ then $\phi \in \Gamma$. In words, Γ is closed under the consequence relation if every formula that can be derived from the formulas in Γ is already in Γ. The smallest S-theory will be of course the set of theorems of S, and, as can be easily seen, the set of theorems of S is included in every S-theory. In what follows we will use the symbols $\mathbf{T}_1$, $\mathbf{T}_2$, etc to refer to theories, in order to distinguish them from ordinary sets of formulas.

A S-theory $\mathbf{T}$ is *generated* by a set of formulas Θ if, for all ϕ, $\phi \in \mathbf{T}$ iff $\Theta \vdash_S \phi$, that is to say, if we can derive any formula of $\mathbf{T}$ from Θ and no formula that can be derived from Θ is outside $\mathbf{T}$. Given any presentation $P = \langle Ax, Ru \rangle$ of S, the set of theorems of S is generated by (the substitution instances of) the statements in Ax. Given our previous characterization of the presentation of a deductive system, we will use the term 'axiom' only to indicate the statements used in a Hilbert-style presentation, and we will employ the term 'hypothesis' to denote the statements used to generate an S-theory different from the trivial one composed only of theorems. We can have different sets of hypotheses for the same S-theory, and these sets can be partially overlapping or completely disjoint. We will use the term 'presentation of theory $\mathbf{T}$' to refer to a set of hypotheses $\Theta^{\mathbf{T}}$ used to generate $\mathbf{T}$.

2.3 Old and new

How do these concepts relate to Lewis'? What we called deductive system has no counterpart in Lewis' account, probably because of the fact that he was considering only one logic, classical logic, and thus he had no need to introduce further distinctions. What Lewis terms 'deductive system' is, in our framework, an S-theory. An S-theory is then what corresponds to a scientific theory. By definition, an S-theory **T** is deductively closed, every formulas that can be deduced from those in **T** is already contained in **T**.

Furthermore, an S-theory is axiomatizable in the sense that it can be generated by a set of hypotheses Θ. We have thus recovered most of Lewis' original idea of a deductive systems as "deductively closed, axiomatizable sets of true sentences". Is there a sense in which an S-theory is a set of *true* sentences?

The answer to this question is: no, unless we take some semantic considerations into account. These would add another layer to our framework. For the rest of this article we will remain at the level of the syntax, running the risk of oversimplification, and leave the semantic side to be developed in future work.

Let us summarize what we have defined in this section. In the framework here presented a scientific theory is composed of the following ingredients:

1. a language L, composed of a signature Σ, a set Con of connectives with the relative maps and some auxiliary symbols.
2. a deductive system S, defined by a consequence relation on the set of formulas generated by L.
3. a set of hypothesis Θ.

A concrete example of a scientific theory presented in a similar fashion can be found in "Axiomatic Foundations of Classical Particle Mechanics" by McKinsey, Sugar and Suppes (McKinsey, Sugar, and Suppes 1953).

2.4 The mathematical apparatus of scientific theories

I have so far ignored the mathematical apparatus employed by many scientific theories. How does mathematics fit into the picture just described? The answer is: we treat mathematical theories as theories in a model-theoretic sense and we add them to the other hypotheses. Therefore, if a scientific theory **T** is using a particular piece of mathematics, an axiomatization of the mathematical notions employed in **T** will be included in the set of hypotheses $\Theta^{\mathbf{T}}$. If, for example, a scientific theory uses real numbers to represent some parameters, we will insert in the mathematical hypotheses an axiomatization of the arithmetic of real numbers.[10]

In this respect it is worth noting that to be able to axiomatize certain mathematical theories we may require a language rich enough to formulate the axioms ('mathematical hypotheses' in our terminology) and a deductive system powerful enough to deduce the desired theorems (some mathematical theories may require second order logic, for example).[11] As a consequence, because of the

[10] As, for example, the one in (Tarski 1994, p. 205).

[11] For a thorough discussion of this matter see (Parsons 2010).

mathematics they employ, some scientific theories cannot even be formulated without assuming a core vocabulary and some kind of minimal deductive power.

The advantage of this account of mathematics is an extreme flexibility: we can tailor the mathematical notions to the need of a scientific theory and study what happens when we modify such notions or their axiomatization (see Subsections 3.1.1 and 3.2 for the implications for simplicity and strength). Moreover, without any specific commitment to the content of the mathematical and non-mathematical hypotheses, we can reasonably hope to describe both the highly-formalized scientific theories, where mathematics is pervasive and there are few non-mathematical hypotheses, and the non-formal scientific theories, where very few mathematical assumptions will be coupled with many non-mathematical hypotheses. Another point worth making is that in this account there is no syntactic characteristic to distinguish between mathematical and non-mathematical hypotheses, in the sense that both are treated as formal statements (maybe the former are more heavily formalized than the latter).

3 Redefining the Core Notions

Having defined a scientific theory as an S-theory, I now turn to the discussion of simplicity and strength. Before analyzing how a theory can be simpler or stronger than another one, however, there is an important observation to make. The comparison between two theories is meaningful, I believe, only if these theories are about overlapping domains of events. To explain this with an example, if I am interested in the laws of nature governing the electromagnetic phenomena I will consider theories that model this kind of phenomena, not Population Biology. This means that at least a naive idea of the intended semantics of our theories is needed if we want to avoid useless comparisons between unrelated theories.

With this in mind and the aid of the framework just defined, let us now turn to simplicity, strength and balance, in this order. In what follows I will intend all the relations in their weak version, that is, we will use the terms 'subset' as short for 'subset or equal', 'less' for 'less or equal in number', and so on.

3.1 Simplicity

We start analyzing simplicity by having a closer look at Lewis' formulation.

3.1.1 Conceptual Simplicity

In the Lewisian model a theory $\mathbf{T}_1$ is simpler than a theory $\mathbf{T}_2$ if $\mathbf{T}_1$ has fewer 'axioms' than $\mathbf{T}_2$. In light of the previous discussion, I maintain that this statement is too vague. Translating this definition into the new terminology one obtains two definitions:

1. $\mathbf{T}_1$ is simpler than $\mathbf{T}_2$ if $\mathbf{T}_1$ has fewer axioms than $\mathbf{T}_2$.

2. $\mathbf{T}_1$ is simpler than $\mathbf{T}_2$ if $\mathbf{T}_1$ has fewer hypotheses than $\mathbf{T}_2$.

depending on how one interprets Lewis' term 'axiom'. There are two observations to make. The first one is that 1 is arguably in contrast with other notions of simplicity. Consider two S-theories $\mathbf{T}_1$ and $\mathbf{T}_2$ with the same language, the same deductive system and the same hypotheses. The difference between $\mathbf{T}_1$ and $\mathbf{T}_2$ lies in the presentation of the deductive system S: the presentation in $\mathbf{T}_1$ has, say, 3 axioms and 2 inference rules; the presentation in $\mathbf{T}_2$ has 10 axioms and 2 inference rules. As can be easily inferred, the derivations of theorems in $\mathbf{T}_1$ will be generally longer than the derivations in $\mathbf{T}_2$, for the derivations in $\mathbf{T}_1$ will require multiple uses of the same axioms to obtain lemmas that can be easily derived in $\mathbf{T}_2$. If $\mathbf{T}_1$ has fewer axioms then derivations in $\mathbf{T}_1$ are more complicated from a computational point of view (see Subsection 3.1.3). The second observation concerns the second definition of simplicity, *conceptual simplicity* from now on. We can have two versions of conceptual simplicity, a *ceteris paribus* one and a general one:

Definition 1 (*Ceteris paribus conceptual simplicity* (CPCS)) *For every pair of theories $\mathbf{T}_1$ and $\mathbf{T}_2$ sharing the same language L and the same deductive system S, we define:*

$$\mathbf{T}_1 \textit{ is simpler}_{CPCS} \textit{ than } \mathbf{T}_2 \textit{ if } \mathbf{T}_1 \textit{ has fewer hypotheses than } \mathbf{T}_2.$$

Definition 2 (General conceptual simplicity (GCS)) *For every pair of theories $\mathbf{T}_1$ and $\mathbf{T}_2$:*

$$\mathbf{T}_1 \textit{ is simpler}_{GCS} \textit{ than } \mathbf{T}_2 \textit{ if } \mathbf{T}_1 \textit{ has fewer hypotheses than } \mathbf{T}_2.$$

As can be easily seen, CPCS is just GCS restricted to theories sharing the same language and deductive system. In its domain of applicability CPCS is an effective measure of simplicity, but such domain is extremely narrow and CPCS cannot be regarded as more than a limiting case. GCS, on the other hand, is defined on every pair of theories, and it is probably the closest to (our interpretation of) Lewis' relation of simplicity. Notably, 'having fewer hypotheses' does not mean that the first set of hypotheses is included in the other as a subset. Substituting the condition of set-theoretical inclusion to the condition on the number of hypotheses one obtains two different relations: CPCS* and GCS*. By the definition of S-theory, CPCS* is nothing more than the relation of inclusion between different set of hypotheses generating the same theory (that is, it is applicable only if $\mathbf{T}_1$ and $\mathbf{T}_2$ coincide). An interesting version of GCS* is:

Definition 3 (Mathematical Simplicity (MS)) *For every two theories* $\mathbf{T}_1$ *and* $\mathbf{T}_2$*:*

$$\mathbf{T}_1 \textit{ is simpler}_{MS} \textit{ than } \mathbf{T}_2 \textit{ if } MH(\mathbf{T}_1)) \textit{ is a subset of } MH(\mathbf{T}_1).$$

where $MH(\mathbf{T}_1))$ and $MH(\mathbf{T}_2))$ denote the set of mathematical hypotheses of $\mathbf{T}_1$ and $\mathbf{T}_2$ respectively.

The reason why a small number of hypotheses is preferable is quite clear: a compact theory is easier to handle and to understand.

Nevertheless, somebody may wonder why the number of hypotheses should be an indicator of simplicity in the first place. The obvious objection is: given a language with conjunction, it is possible to conflate finitely many formulas into one by taking the conjunction of them (even infinitely many, if the language has infinitary conjunctions). This of course makes the counting of hypotheses an irrelevant matter. This however is not a problem, for two reasons. The first, of a pragmatic flavour, is simply that there are no theories with hypotheses where the conjunction is the main connective. The second is that, even if we want to avoid pragmatic considerations, it is possible to write a simple computer programs that, in counting the number of hypotheses, checks whether the hypotheses have a conjunction as outer connective. If this is the case, the program consider the subformulas as distinct hypotheses, and restarts the counting (and the check). As long as we have hypotheses made of finitely many symbols, the program will output the correct number of axioms, despite of conjunctions.

Nevertheless, the number of hypotheses is just one of the components of a theory, and we should also consider the role of languages and deductive systems.

3.1.2 Expressive simplicity

As far as the language is concerned, we can compare two theories in terms of the expressive power of their signatures, of their *expressive simplicity*. Let us explain this with an example. Consider two theories $\mathbf{T}_1$ and $\mathbf{T}_2$ such that in both their signatures there is a sort A. In the language of $\mathbf{T}_1$ there is a symbol for a constant of sort A, while in the language of $\mathbf{T}_2$ there is no such symbol, and thus to refer to the same object we have to use a paraphrase like "the object of sort A satisfying conditions x, y, etc". The same argument can be applied to every other symbol of the signature: to function symbols ("the function of type $A\ldots$ satisfying conditions x, y, etc") and to relation symbols ("the relation of type $A\ldots$ satisfying conditions x, y, etc").

The signature of $\mathbf{T}_2$ is simpler in the sense that it has less symbols and that some symbols of $\mathbf{T}_1$ can be substituted by a combination of symbols of $\mathbf{T}_2$. This feature can be important if we want to minimize the number of primitives for foundational purposes. The signature of $\mathbf{T}_1$ is simpler in the sense that is less cumbersome, instead of repeating a long list of symbols we can just employ a shorter expression. This can make the difference, for example, from a didactic perspective or for computational complexity. We have here two conflicting notions of simplicity.

Definition 4 (Expressive Simplicity with Less Symbols (ESLS)) *For every two theories* $\mathbf{T}_1$ *and* $\mathbf{T}_2$*:*

$$\mathbf{T}_1 \textit{ is simpler}_{ESLS} \textit{ than } \mathbf{T}_2 \textit{ if } \mathbf{T}_1 \textit{ has less symbols than } \mathbf{T}_2$$

Definition 5 (Expressive Simplicity with More Symbols (ESMS)) *For every two theories* $\mathbf{T}_1$ *and* $\mathbf{T}_2$*:*

$$\mathbf{T}_1 \textit{ is simpler}_{ESMS} \textit{ than } \mathbf{T}_2 \textit{ if } \mathbf{T}_1 \textit{ has more symbols than } \mathbf{T}_2$$

Explicitating a particular kind of symbols in these definitions, respectively sort, function and relation symbols, we have three more specific versions of ESLS and ESMS. Along these lines, the importance of ESLS and ESMS can be weighed relatively to the symbols under examination: we may want a symbol with a pivotal role in our theory, say, the constant representing the speed of light, to be included in the signature, while a conceptually subordinate symbol may be defined in terms of others.

3.1.3 Computational Simplicity

It is also possible to find notions of simplicity connected with the deductive system of a theory. Consider for example the following case. Given a set of formulas Γ regarded as true, say, a set of formulas representing empirical observations or some important theorems, a theory $\mathbf{T}_1$ may be judged simpler than a theory $\mathbf{T}_2$ if the derivations of the formulas in Γ in $\mathbf{T}_1$ are 'simpler' than the corresponding derivations in $\mathbf{T}_2$.

But how can a derivation be simpler than another one? Before examining possible candidates of *computational simplicity*, one has to qualify two points. First, there are two variables to consider: which deductive system is used and how it is presented. A 'stronger' deductive system, one which is an extension of another one, for example, may produce simpler proofs (see below for examples of what this can mean). A more compact presentation, one employing fewer axioms or inference rules, will usually determine more complex derivations. Second, as far as computational simplicity is concerned, the choice of connectives has to be considered as part of the presentation of a deductive system. A wide set of connectives without the relative axioms or inference rules (say, having the symbol of conjunction but only axioms and inference rules for the symbol of entailment) cannot enhance the simplicity of derivations and, vice versa, axioms and inference rules can be used only in the presence of the relative connective symbol. This is why the choice of connective symbol is relevant for computational simplicity and not only for expressive simplicity. Here are two proposals for computational simplicity:

Definition 6 (Computational Simplicity in Length (CSL)) *For every two theories* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *and for every set of formulas* Γ*:*

$\mathbf{T}_1$ *is simpler*$^{\Gamma}_{CSL}$ *than* $\mathbf{T}_2$ *if all the derivations of the formulas in* Γ *in* $\mathbf{T}_1$ *are shorter than those in* $\mathbf{T}_2$

To be able to compare the lengths of proofs we have to introduce a measure of such length (usually the number of lines).

Definition 7 (Computational Simplicity in Time (CST)) *For every two theories* $\mathbf{T}_1$ *and* $\mathbf{T}_2$*, for every set of formulas* Γ *and given a suitable automated theorem prover (a computer program that produces derivations), we define:*

$\mathbf{T}_1$ *is simpler*$^{\Gamma}_{CST}$ *than* $\mathbf{T}_2$ *if all the derivations of the formulas in* Γ *from the hypotheses in* $\mathbf{T}_1$ *take less time than those in* $\mathbf{T}_2$

Depending on the prover employed, this may require that $\mathbf{T}_1$ and $\mathbf{T}_2$ share the same deductive system. As long as Γ consists of a single formula, we can apply CSL and CST without worries. But if Γ contains two or more formulas one could have problems of applicability. Consider a case where $\mathbf{T}_1$ is simpler$^{\Gamma*}_{CSL}$ than $\mathbf{T}_2$ and $\mathbf{T}_2$ is simpler$^{\Gamma+}_{CSL}$ than $\mathbf{T}_1$, where $\Gamma*$ and $\Gamma+$ are two disjoint subsets of Γ. In this case CSL cannot be applied relatively to Γ (an analogous argument can be made for CST). To overcome this impasse and define a universally applicable version of CSL (CST respectively) we may define a total measure of length (respectively time) for the derivations of the formulas in Γ and then compare the total measure in $\mathbf{T}_1$ with the total measure in $\mathbf{T}_2$ instead of comparing derivations pairwise. This approach leads to a generalized version of CSL (respectively CST).

It remains to say why these notions of simplicity are interesting candidates. A common agument can be made for CSL and CST. It is essentially an optimization argument: given any application of a theory (for example checking whether some formulas follow from the theory or not) we prefer the theory that requires less effort to be used. Indeed, the fact that a theory is computationally expensive can be a reason to change or improve the theory.

3.2 Strength

For Lewis a theory is stronger than another if it has more informational content (Lewis 1999, p. 41). If we interpret the informational content of a theory $\mathbf{T}$ as all the formulas that can be derived from the hypotheses of $\mathbf{T}$ we have that, by definition of S-theory as a deductively closed set of formulas, the informational content of $\mathbf{T}$ coincides with $\mathbf{T}$. If one sticks to this interpretation it is possible to formulate strength as:

Definition 8 (General Strength (GS)) *For every two theories* $\mathbf{T}_1$ *and* $\mathbf{T}_2$*:*

$\mathbf{T}_1$ *is stronger*$_{GS}$ *than* $\mathbf{T}_2$ *if* $\mathbf{T}_2$ *is a subset of* $\mathbf{T}_1$

GS is interesting because it encodes the fact that we can reduce one theory to another, that is, we can prove all the statements of the first one inside the second one. There are cases, however, where GS cannot be applied. A sets of formulas can be included in another only if they share the same language, or the language of the bigger set is an extension of the other one. Another approach could be the following. Given a set of true formulas Γ, say, the formulas representing the observations made, the informational content of theory $\mathbf{T}$ is the portion of Γ that is derivable from the hypotheses of $\mathbf{T}$, that is, the intersection between Γ and $\mathbf{T}$. Of course the formulas in Γ have to refer to the shared part of the intended domain of interpretation, otherwise one of the two theories will be weaker a priori. We then have:

Definition 9 (Informational Strength (IS)) *For every two theories* $\mathbf{T}_1$ *and* $\mathbf{T}_2$*, and given a set of formulas* Γ*:*

$\mathbf{T}_1$ *is stronger*$^{\Gamma}_{IS}$ *than* $\mathbf{T}_2$ *if the informational content of* $\mathbf{T}_1$ *relative to* Γ *is bigger than that of* $\mathbf{T}_2$

where by bigger I mean cardinality-wise. One can of course restrict this notion substituting 'is bigger than' in the definition with 'includes' obtaining IS*. Obviously, IS* entails IS for every Γ.

We could also relate the notion of strength to that of deductive system:

Definition 10 (Computational Strength (CS)) *For every two theories* $\mathbf{T}_1$ *and* $\mathbf{T}_2$*:*

$\mathbf{T}_1$ *is stronger*$_{CS}$ *than* $\mathbf{T}_2$ *if* $\vdash_{\mathbf{T}_2}$ *is a subset of* $\vdash_{\mathbf{T}_1}$

In other words, the deductive system of $\mathbf{T}_1$ is stronger$_{CS}$ than that of $\mathbf{T}_2$ if in $\mathbf{T}_1$ we can derive every formula derivable in $\mathbf{T}_2$ and some more. Notably, if $\mathbf{T}_1$ and $\mathbf{T}_2$ share the same set of hypotheses then CS implies GS and IS* for every Γ. CS holds even though $\mathbf{T}_1$ and $\mathbf{T}_2$ do not share the language, as the language of, say, $\mathbf{T}_1$ can be an extension of that of $\mathbf{T}_2$.

Along the same lines of CS one can introduce a notion of strength connected with the mathematical apparatus of theories. A first option can be the inverse relation of MS:

Definition 11 (Mathematical Strength (MSt)) *For every two theories* $\mathbf{T}_1$ *and* $\mathbf{T}_2$*:*

$\mathbf{T}_1$ *is stronger*$_{MSt}$ *than* $\mathbf{T}_2$ *if* $MH(\mathbf{T}_2)$ *is a subset of* $MH(\mathbf{T}_1)$

We have here a straightforward example of the conflict between a relation of strength and a relation of simplicity: if $\mathbf{T}_1$ is simpler$_{MS}$ than $\mathbf{T}_2$ then $\mathbf{T}_2$ is stronger$_{MSt}$ than $\mathbf{T}_1$. However, this is not the case in general for the notions that we defined, for example Expressive Simplicity is independent from Mathematical Strength. Hence the trade-off between simplicity and strength mentioned by Lewis is a consequence of particular selections of notions of simplicity and strength.

Alternatively, one can impose a further condition to have a more informative relation:

Definition 12 (Strict Mathematical Strength (SMS)) *For every two theories* $\mathbf{T}_1$ *and* $\mathbf{T}_2$*:*

$$\mathbf{T}_1 \textit{ is stronger}_{SMS} \textit{ than } \mathbf{T}_2 \textit{ if } MH(\mathbf{T}_2) \textit{ is a proper subset of } MH(\mathbf{T}_1)$$

This last relation might be appealing if we think that a particular mathematical theory is essential to model a certain class of phenomena, say, Hilbert spaces to model Quantum phenomena, and we want to draw a distinction between theories that employ such mathematical machinery and theories that do not.

3.3 Balance

Depending on the notions of simplicity and strength adopted, we can define balance in many ways. Following the characterization of simplicity and strength as binary relations, I will treat balance as a binary relation as well, that is to say, I will consider relative balance. In the presence of some absolute measures of simplicity and strength, absent in the present work, one may attempt a definition of the absolute balance of a theory.

As can be easily checked, apart from SMS all the relations defined are preorders in their respective domain of applicability, that is, they are reflexive and transitive. With this in mind, let us sketch two general procedures to define the balance. Suppose we have a set of theories to evaluate and a collection of relations of simplicity and strength.

The first procedure, of a qualitative nature, consists of aggregating the orderings of the set of theories produced by the chosen relations. Formally, this means that given n orderings $R_1, \ldots, R_n$ we want to have a procedure to obtain a single ordering R. The top theory/theories according to this last relation will be the best system(s). Of course, depending on how we aggregate these orderings we will obtain different outcomes. One first question to pose in this respect is: are all orderings equally relevant or do we regard some criteria as privileged?

A mathematical environment where such an aggregation procedure can be studied is provided by Social Choice.[12] To make an example, in this framework the condition encoding the idea that all orderings must be equally relevant is called anonimity (invariance of the aggregator under the permutations of the input orderings). In this context, given two theories $\mathbf{T}_1$ and $\mathbf{T}_2$ and k relations corresponding to the equally relevant selection criteria, we may say that $\mathbf{T}_1$ is better than $\mathbf{T}_2$ if $\mathbf{T}_1$ is preferable according to $k/2 + 1$ relations. The extent to which results and techniques of Social Choice can be applied to the present case will be explored in future work.

[12] For a standard reference in the field see (Gaertner 2009).

The second procedure involves the definition of quantitative measures relative to the chosen relations. If, say, $\mathbf{T}_1$ is simpler$_{GCS}$ than $\mathbf{T}_2$ we could take the difference between the number of hypotheses in $\mathbf{T}_2$ and the number of hypotheses in $\mathbf{T}_1$ as a number representing how much simpler $\mathbf{T}_1$ is compared to $\mathbf{T}_2$. By similar methods, counting or using percentages, one may associate a function to each relation in order to evaluate the relative degree of simplicity or strength. If this attempt succeeds one can then use these functions to construct an algorithm able to analyze the set of theories, apply such functions and combine the results to find the theories that score the best combination according to the chosen relations of simplicity and strength. To continue the example above, we could assign weights $n_1, \ldots, n_k$ to the k relations and say that the score of $\mathbf{T}_1$ is the sum of the weights of the relations in which $\mathbf{T}_1$ is preferable over $\mathbf{T}_2$. We could then conclude that $\mathbf{T}_1$ is better than $\mathbf{T}_2$ if $\mathbf{T}_1$ has a higher score.

Before concluding, we make three final remarks. The first is that the choice of the collection of relations of simplicity and strength does not influence the balance function just by changing the arity of its input. In the second methodology a particular choice of relations might change the internal structure of the algorithm. For example, if we employ General Strength we might want the algorithm to check this relation first, to know whether one theory is reducible to the other. The second remark is that in both cases if the chosen relations cannot be applied to the set of theories, because theories do not share enough features for the relations to be applied, we could not find any best system. The third remark concerns the viability of the two methodologies. Both of them are applicable only if the chosen relations are decidable. If they are not, then in the first case we might not get the orderings at all, and in the second case the algorithm may not terminate.

3.4 Conclusion

Let us draw some conclusions. In light of the formal analysis outlined and of the examples offered, I argue that the aforementioned framework is appropriate for a precise characterization of the notions of simplicity, strength and balance. Moreover, I believe that the plurality of definable notions of simplicity (respectively, strength and balance) casts doubt on Lewis' reliance on a single concept and demands for a more comprehensive discussion. Simplicity, strength and balance are, I think, multifaceted ideas, and the search for a unique characterization could be misleading. This of course does not imply that such notions have to be vague, as the present work showed.

Indeed we have alternative versions of BSA depending on

1. which relations of simplicity and strength we use

2. how do we aggregate them to obtain the balance

It is already hard to reach a consensus on the first item. For an experimental physicist, interested in testability and implementations, theories may be compared with an eye for their computational features. A philosopher, on the other hand, could think that the best theory is one with few primitives.

The advantage of our framework, as long as it is considered tenable, is that now we can look at specific, well defined candidates for relations of simplicity and strength. Likewise, we can design and analyze procedures to obtain the balance. This means that the discussion about item 1 and 2, although still philosophical in nature, is now more formally grounded.

References

Armstrong, David M. (1983). *What is a Law of Nature?* Cambridge: Cambridge University Press.

Bird, Alexander (1998). *Philosophy of Science*. London: UCL Press.

— (2008). “The Epistemological Argument against Lewis’s Regularity View of Laws”. In: *Philosophical Studies* 138, pp. 73–89.

Carnap, Rudolf (1937). *The Logical Sintax of Language*. Trans. by A. Smeaton. London: Routledge 2001.

Chang, Chen Chung and H. Jerome Keisler (1990). *Model Theory*. 3rd ed. Studies in Logic and the Foundations of Mathematics 73. Amsterdam: North-Holland.

Cohen, Jonathan and Craig Callender (2009). “A Better Best System Account of Lawhood”. In: *Philosophical Studies* 145, pp. 1–34.

Da Costa, Newton and Steven French (2000). “Models, Theories, and Structures: Thirty Years On”. In: *Philosophy of Science (Proceedings)* 67, S116–S127.

Font, Josep M., Ramon Jansana, and Don Pigozzi (2003). “Survey of Abstract Algebraic Logic”. In: *Studia Logica* 74 (Special issue on Abstract Algebraic Logic, Part II), pp. 13–97. With an update in 2009, 91: 125-130.

Gaertner, Wulf (2009). *A Primer in Social Choice Theory*. Oxford: Oxford University Press.

Jaeger, Lydia (2002). “Humean Supervenience and Best-System Laws”. In: *International Studies in the Philosophy of Science* 16.2, pp. 141–155.

Johnstone, Peter T. (2002). *Sketches of an Elephant: A Topos Theory Compendium*. Vol. 2. Oxford: Oxford University Press.

Lewis, David (1973). *Counterfactuals*. Oxford: Blackwell.

— (1986). *Philosophical Papers*. Vol. 2. New York: Oxford University Press.

— (1994). “Humean Supervenience Debugged”. In: *Mind* 103, pp. 473–490.

— (1999). *Papers in Metaphysics and Epistemology*. Cambridge: Cambridge University Press.

McKinsey, John C.C., A.C. Sugar, and Patrick Suppes (1953). “Axiomatic Foundations of Classical Particle Mechanics”. In: *Journal of Rational Mechanics and Analysis* 2.2, pp. 253–272.

Mumford, Stephen (2004). *Laws in Mature*. London: Routledge.

Parsons, Charles (2010). “Some Consequences of the Entanglement of Logic and Mathematics”. In: *Reference and Intentionality: Themes from Føllesdal*. Ed. by W.K. Essler and M. Frauchiger. Frankfurt: Ontos Verlag.

Psillos, Stathis (2002). *Causation and Explanation*. Chesham: Acumen.

Robert, John (1999). "“Laws of Nature” as an Indexical Term: A Reinterpretation of Lewis's Best-System Analysis". In: *Philosophy of Science* 66 (Proceedings), S502–S511.

Tarski, Alfred (1944). "The Semantic Conception of Truth and the Foundations of Semantics". In: *Philosophy and Phenomenological Research* 4.3, pp. 341–376.

— (1994). *Introduction to Logic and to the Methodology of Deductive Sciences*. 4th ed. Oxford: Oxford University Press.

Van Fraassen, Bas (1989). *Laws and Symmetry*. Oxford: Clarendon Press.