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Some proposals for the set-theoretic foundations of category theory

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Abstract. The problem of finding proper set-theoretic foundations for category theory has challenged mathematician since the very beginning. In this paper we give an analysis of some of the standard approaches that have been proposed in the past 70 years. By means of the central notions of class and universe we suggest a possible conceptual recasting of these proposals. We focus on the intended semantics for the (problematic) notion of large category in each proposed foundation. Following Feferman (2006) we give a comparison and evaluation of their expressive power.

Keywords. Category Theory, Set Theory, Foundations..

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1 A problem of size

> [...] Thus, category theory is not just another field whose set-theoretic foundation can be left as an exercise. An interaction between category theory and set theory arises because there is a real question: what is the appropiate set-theoretic foundation of category theory?
>
> Andreas Blass[1]

It is common to date the birth of category theory to the publication of Eilenberg and Mac Lane's paper,[2] *A general theory of natural equivalences*. Already in this pioneering work, we can find a first analysis of some foundational issues concerning the raising theory. In fact Eilenberg and Mac Lane dedicate an entire paragraph to discuss some foundational problems of the set-theoretical interpretation of their theory. Here is the beginning of this paragraph:[3]

> We remarked in §3 that such examples as the "category of all sets", the "category of all groups" are illegitimate. The difficulties and antinomies here involved are exactly those of ordinary intuitive Mengenlehre; no essentially new paradoxes are apparently involved. Any rigorous foundation capable of supporting the ordinary theory of classes would equally well support our theory. Hence we have chosen to adopt the intuitive standpoint, leaving the reader free whatever type of logical foundation (or absence thereof) he may prefer.

The two authors immediately recognise the peculiarity of the constructions involved in their theory and offer a first simple diagnosis: since there is nothing new under the sun, just old well-known paradoxes, it is sufficient to give back these issues to the field they belong, i.e. set theory. Despite the apparent haste to dismiss the matter, what follows the above mentioned paragraph can be seen as the first concrete attempt to solve the problem: after having discussed some technical issues, the two mathematicians suggest a possible development of category theory inside the framework of the theory of sets and classes in the style of von Neumann, Bernays and Gödel's set theory (NBG). Before entering into the details of this and other proposals, it is important to focus on what is the problem. A good starting point is given by a critical analysis of the role played by the notion of **size** in category theory. Indeed, with the exception of set theory, it is difficult to find other mathematical fields where the notion of size plays such

[2] S. Eilenberg (1945).
[3] S. Eilenberg (1945), p. 246.

a central role. On the other hand, in category theory the distinction between **small categories** and **large categories** represents an important and inescapable dichotomy raised at the very beginning of any reasonable introduction to the subject. Nevertheless it is usual to get rid of this question as soon as possible and the working mathematician who uses category theory is therefore reluctant to deepen the analysis of the foundations of the theory. The following dialogue[4] is intended to parody this situation:

Dialogue 1.

TORTOISE: Hi Achilles, how are you? You have disappeared for a while, what have you been up to?

ACHILLES: My dear little Tortoise, you won't believe it, but I started studying some *abstract nonsense*. And, let me say that I found in it much more sense than is usually said.

TORTOISE: Good Achilles, I see you are not losing the habit to challenge your mind. I also have tried to give meaning to that bunch of arrows some time ago... now, I can just remember the definition of a category. Let me take the opportunity to ask you something that has bothered me since that time. Can you tell me what people mean with the term *large category*?

ACHILLES: Oh, my sweet little Tortoise, I know what you are driving at... you want to cheat me with the old story of the barber undecided if he shaves himself or not... this time I won't fell for it. The matter is simple: a large category is one whose collection of morphisms is a *proper class*.

TORTOISE: Then, let me bother you with my usual reasoning. The natural question to pose now is: what do you mean by proper class?

ACHILLES: Well, I'll be polite and I won't escape your innocent inquisition. I will call a proper class a collection which is not a set.

TORTOISE: It's not exactly a definition, but I'll give you that. I believe you already know what I am going to ask next...

ACHILLES: Let's see. Usually you don't have so much imagination. The only new term I introduced in our dialogue is set. I hope you don't want to ask me what is a set...

[4]The characters of this invented dialogue have been inspired by the dialogues in Hofstadter (1979).

TORTOISE: Exactly Achilles: less fantasy and more pedantry is the recipe of my philosophy…

ACHILLES: Ok. Let me surprise you. I have a new definition: a set is an object of the category Set, whose object are sets and whose morphisms are functions.

TORTOISE: Mmh…, you are right Achilles, you always surprise me…I am afraid you lost your way in an abstract nonsense…

Clearly positions like Achilles' one are unsatisfying from every possible point of view: mathematical, logical and philosophical. A proper category theorist, probably, would have preferred to answer Tortoise's question, "what is a set", saying "it's an object of a well-pointed topos with a natural number object and which satisfies the axiom of choice". Since this answer costs much more effort than trying to understand the problem, it is important to clarify what we mean by the problem of set-theoretic foundations of category theory, in such a way that also Achilles can understand why his position is not defensible.

It is an empirical fact that, to a great extent, mathematics can be formalized in set theory: a rather common choice for this set-theoretic "codification" is represented by the axioms of Zermelo Fraenkel's set theory with the axiom of choice (ZFC). For example, we can imagine to present group theory, algebraic topology or functional anaysis with the language only of set theory: objects of these theories can be described as sets whose properties can be derived from set-theoretic axioms. Following Blass,[5] it is therefore natural to ask in what sense category theory is an exception to this phenomenon. Why can't we leave this codification as a routine exercise?

As we have already observed, at the root of category theory lies the important *small/large distinction*. When doing category theory some of the objects and constructions that we deal with are (and have to be) essentially large. One of the first problem we meet if we regard this object from a set- theoretic perspective is to find an adequate encoding for large categories such as the category of all sets (*Set*) or the category of all groups (*Grp*). These categories are built having in mind essentially large collections and cannot be treated simply as sets.[6] This is not the only problem. The following list resumes some of the main issues that are essential to develop category theory.[7] In every reasonable foundational framework[8] we want to be able to:

[5] See the quotation at the beginning of this section.

[6] The argument is well known. A possible way to present it is the following: if the collection of all sets, V, was a set, then the collection of all its subsets, $\wp V$, would be a set included in V, contradicting Cantor's theorem.

[7] Compare with Feferman (2006) pp. 2–3.

[8] We use framework as synonym of metatheory or foundational system.

(A) form the category of every structure of a given type. Some elementary examples are: *Set*, *Grp* and *Top*;

(B) perform some basic set-theoretic constructions over an arbitrary category;

(C) form the category of all the functors between two arbitrary categories.

If we are specifically interested in set-theoretic foundations for category theory we would also like to be able to

(D) decide the consistency of these systems with respect to some accepted system of set theory.

It is worth mentioning that, beyond the concept of "large category" (requirement A), there are several different notions that rely on the same concept (*locally small category*, *small limits*, etc.). The frameworks should be expressive enough to make sense of each of these.

As we will see the choice of a specific foundational system will affect substantially the fulfilment of these requirements.

The next section gives an overview of the foundational proposals that we will consider in the rest of the paper.

2 Set-theoretic and other proposals: a retrospective.

As already noted, debates about foundations of category theory started with the very introduction of the notion of category. The rapid development of the theory and the ubiquity of categorical notions in different mathematical fields have brought these foundational issues to the attention of several mathematicians.

In the sequel we will consider some standard set-theoretic approaches to the problem of foundations of category theory. It is important to keep in mind that set theory is just *one* possible approach. Even among the set-theoretic frameworks, we won't be able to cover exhaustively all those proposed in the past, for example, Feferman's proposal to use Quine's set theory, *New Foundations*.[9] The question of what the *proper* set-theoretic foundation of category theory is can be misleading. We could argue that category theory, as any other mathematical subject, does not need any foundation either for its own internal development, or for understanding it. Nevertheless, once we have raised the question, we find that different solutions are at our disposal. As we will see none of the set-theoretical proposals we will consider definitely solve the problem. However, we stress that to a great extent each of these proposals is expressive enough to cover most of the cases of interest for the "working mathematician".

An important part in the debate of foundations of category theory that deserves a treatment on its own, is the possibility to regard category theory itself

[9]The interested reader should consult Feferman (2006).

as a foundational theory. The idea to consider category theory as a universal language capable of interpreting the entire mathematical edifice has been firstly proposed by Lawvere in the mid 60s. His research has led to a purely categorical description of the category *Set*. Nowadays, after his influential paper,[10] it is common to refer to these axioms with the acronym ETCS: Elementary Theory of the Category of Sets.

From a philosophical perspective, the project of Lawvere is intertwined with what has been called *categorical structuralism*.[11] As recent debates have shown, progress is impossible without a preliminarily agreed understanding of what is meant by the use of adjectives "structural" and "foundational" in this context.[12] Close to categorical structuralism, but not coinciding with Lawvere's position, is the idea to regard category theory as a foundation because of its *unifying character*. This position emerges for example in Marquis[13] and has recently been supported by some novel results discovered in topos theory.[14]

We finally mention a recent area of research that investigates set theory from a novel categorical perspective: Algebraic Set Theory (AST). The main goal of AST is to give a uniform categorical description for set-theoretical formal systems. Without addressing directly any foundational issues, AST focuses on bringing to light algebraic aspects of these systems by means of category theory.[15]

We can now focus on the organisation of the foundational proposals that we consider. The frameworks we will treat are the following:

- an approach internal to ZFC,
- NBG and MK,[16]
- Grothendieck's universes,
- Mac Lane's proposal,
- Feferman's proposal.

The first two set theories have in common the idea of using the notion of *class* to interpret the notion of *size* arising in category theory. The other three, instead, make use (in a more or less explicit way) of the notion of *universe* in order to better approximate the distinction *small/large*. Inspired by Shulman (2008), we suggest a possible recast of these proposals by means of these central notions.

[10]Lawvere (2005)

[11]See for example Awodey (1996), McLarty (2004).

[12]The interested reader should consult Hellman (2003) and Awodey (2004).

[13]See Marquis (2009).

[14]See Caramello (2010).

[15]A standard reference for AST is the book Joyal 1995. For a complete bibliography the reader should visit http://www.phil.cmu.edu/projects/ast/.

[16]MK is the acronym for Morse-Kelley set theory.

Before giving the details of these possible solutions we recall, in the next section, some specific examples of theorems "sensitive to the mathematical framework".

3 Examples

To give an idea of the ubiquity of notions of size in category theory we recall some basic results and definitions where these concepts play an important role[17]

Definition 1 (locally small category). *A category* $\mathbb{C}$ *is called locally small if, given two objects, a and b, the collection of morphisms between them,* $Hom_{\mathbb{C}}(a, b)$, *is small.*

If a category $\mathbb{C}$ is locally small then there exists the Hom-functor:

$$Hom_C : C^{op} \times \mathbb{C} \to Set.$$

Examples of locally small categories are: *Set*, *Grp* and in general all the categories built from "sets-with-structure". Given two locally small categories $\mathbb{C}$ and $\mathbb{D}$, the category of functors between them, $\mathbb{D}^{\mathbb{C}}$, is not in general, locally small.

A central notion in category theory is that of complete category: also in this case instances of the notion of size are explicitly involved.

Definition 2 (complete category). *A category* $\mathbb{C}$ *is said to be* ***complete*** *if every functor* $F : J \to \mathbb{C}$, *whose domain is a small category J, has limit.*

Examples of complete categories are: *Set*, *Grp*, *Rng*, *Comp Haus*. When the category is both small and complete, then it is just a preorder. Actually something stronger holds:

Theorem 2. *If a category* $\mathbb{C}$ *admits limits for any functor* $F : \mathbb{D} \to \mathbb{C}$, *with* $\mathbb{D}$ *any discrete category, then* $\mathbb{C}$ *is a preorder.*

For the proof see Borceaux (1994), proposition 2.7.1. This theorem explains why, in order to have a notion of *completeness* which makes sense for all categories, it is reasonable to ask for limits just for those functors whose domain is a small category J.

Another important theorem which is usually quoted when debating foundational issue in category theory is *Freyd's adjoint theorem*. We briefly recall some definitions which occur in the body of this theorem.

[17] Most of the examples here and in the rest of the paper can be found in Shulman (2008).

Definition 3. *A category is said to be* ***well-powered*** *if each of its objects admits a poset of subobjects.*

Definition 4. *A family* Q *of objects in a category* $\mathbb{C}$ *is called* ***cogenerating*** *if, given two parallel distinct morphisms,* $f \neq g : a \to b$*, there is a morphism* $h : b \to q$ *with* $q \in Q$ *such that* $hf \neq hg$*.*

Theorem 3. *Given a locally small, complete, well-powered, category* $\mathbb{C}$ *endowed with a cogenerating set, and a category* $\mathbb{D}$*, locally small, a functor* $G : \mathbb{C} \to \mathbb{D}$ *has a left adjoint if and only if it preserves small limits.*

For the proof see Lane (1998) ch. 5, par. 8. Note that this theorem relies essentially on some notion of size. If the theorem is expressed just for small categories we obtain the following.

Corollary 4. *Given a complete lattice,* $\mathbb{C}$*, a lattice morphism,* $G : \mathbb{C} \to \mathbb{D}$*, which preserves infima has a left adjoint.*

Clearly this corollary is just a shadow of Freyd's adjoint theorem. The significance of this latter can be appreciated if we think that in some cases this result represents the only device to build an adjunction.

4 Large categories and classes

small = "set" / large = "class"

Classes (more precisely proper classes) arise in classical set theory (ZFC) as those logical formulas without proper citizenship in models of set theory. They are built from set-theoretical formulas by means of unrestricted comprehension, and, even without a proper ontology,[18] they are commonly introduced as a useful device for manipulating formulas they abbreviate. As we are going to see in the next paragraphs, classes represent possible candidates to interpret large categories in a set-theoretical framework.

4.1 An approach internal to ZFC

A possible choice to give meaning to the notion of **large** categories is suggested by the usual convention adopted to introduce **classes** in ZFC. A class in the lan-

[18] They don't have proper ontolgy since they are outside the domain of discourse described by the axioms. Following Quine we can say that classes "do not have being" since they are not values of bound variables.

guage of ZFC is a formal expression of the form $x|\phi(x)$ where ϕ is a formula of the language of ZFC. Every set can be seen as a class (of its elements) but, by Russell's paradox, the converse is not true. We say that a class is a **proper class** if it is not a set.

Example 5. *The class of all sets, V, is defined by the formula*

$$V := \{x|x = x\}.$$

Another well-known proper class is the collection Ω of all ordinals. By the Burali-Forti paradox it cannot be a set.

The idea of this foundational recipe is very simple: we call a category *large* when the collection of its objects is a *proper class*.

One virtue of this approach is to work internally to ZFC: even if we cannot directly manipulate large objects we are still able to work with the properties (logical formulas in the language of ZFC) which define them. In this way we still have the possibility to perform simple basic constructions over large categories: for example if ϕ and ψ are formulas of ZFC we can still form the class of pairs whose first element satisfy ϕ and whose second element satisfy ψ, i.e. we can form the cartesian product of the two categories corresponding to ϕ and ψ.

The real problem of this approach is that ZFC cannot quantify over classes: theorems saying "there is a category such that..." or "for every category..." cannot even be stated in ZFC (one example is given by Freyd's adjoint theorem). Therefore, if we choose this foundational framework, we are led to reformulate most of our theorems as meta-theorems, which seems quite unpleasant from a foundational perspective.

4.2 NBG and MK

The most common set theory which introduces an ontology both for classes and sets is von Neumann, Bernays and Gödel's set theory (NBG). We briefly recall the axioms

(i) *axioms in common with ZFC*: pair, union, infinity, powerset;

(ii) *axioms both for sets and classes*: extensionality, foundation;

(iii) *axiom of limitation of size*: a class is a set if and only if it is not in bijection with the class of all sets V.

(iv) *axiom schema of comprehension*: for every property $\varphi(x)$, without quantifiers over classes, there exists the class $\{x|\varphi(x)\}$.

The system NBG is a conservative extension of ZFC: every sentence, *relative to sets*, which is provable in NBG, is already provable in ZFC. Therefore having

NBG as a foundation does not imply any particular ontological commitment. The differences with ZFC are mainly at a stylistic level.[19] As we mentioned in the first paragraph the use of NBG as a possible foundation for category theory trace back to the original paper of Eilenberg and Mac Lane.[20] The advantage of NBG with respect to ZFC consists essentially in the explicit treatment of classes: several constructions become easier, and, moreover, it is legitimate to quantify over classes. As suggested in Shulman (2008), another interesting feature of NBG consists in the possibility of adopting a form of **global choice**. This, surprisingly, is an easy consequence of the axioms. Consider the following observation due to von Neumann:

Theorem 6. *In NBG, the class of all sets, V, is well-orderable.*

Proof. The class Ω of all ordinals is a proper class and it is well-ordered. By the axiom of limitation of size this class is in bijection with V. This bijection induces a well order on V. □

The fact that V is well-orderable is one of the possible formulations of the axiom of choice for classes; in category theory the possibility to have this large choice is sometimes essential. In fact we are generally assuming it when choosing representatives of universal constructions over large categories. Despite these good points, and the several advantages over the approach internal to ZFC, NBG still presents some problems as a possible foundational framework for category theory: one, for example, is the use of comprehension restricted to formulas not involving classes.[21] A possible solution is then to strengthen the axioms of NBG by allowing for arbitrary quantification in the formulas involved. The resulting theory is known as Morse-Kelley set theory (MK). In this case, however, we have lost conservativity over ZFC, and the theory we end up with is genuinely stronger than ZFC.

In all the cases examined so far, a central problem has still to be overcome: none of these systems allow for the construction of the category of functors between two arbitrary categories. We can form the category of functors from a small category to an arbitrary one,[22] but this construction still remains illegiti-

[19] Historically the interest in this system have been motivated by the search for an equivalent system to ZFC which was finitely axiomatizable.

[20] S. Eilenberg (1945).

[21] When proving a statement $\varphi(n)$ by induction in ZFC or NBG we usually form the set $\{n \in N | \neg\varphi(n)\}$ and then use the fact that N is well-ordered. Since this argument involves an istance of comprehension, it can be carried on into these systems just in case φ does not involve quantifiers over classes.

[22] This is allowed in all the cases examined so far: for example in ZFC a functor $F : \mathbb{C} \to \mathbb{D}$, where $\mathbb{C}$ is a small category, is itself a set by replacement, and therefore the collection of all these functors form a class.

mate when the domain of these functors is a large category. However, to a great extent all these systems are expressive enough to cope with the cases of interest: even if the functor category seems a perfectly reasonable construction which can be performed regardless of size issues, most of category theory can be developed confining our attention to those functors whose domain is a small category. This limitation is consistent with the one on completeness.[23]

In summary, the foundational frameworks considered so far fulfil (with different degrees of approximation) the requirements (A) and (B) (p. 4), but none of them manages to satisfy (C) in its full generality. To sum up relative consistency of these systems (D) we can say that V_α models ZFC if and only if $(V_\alpha, Def(V_\alpha))$[24] models NBG. If α is an inaccessible than $(V_\alpha, V_{\alpha+1})$ models both NBG and MK.

5 Large categories and Universes

$small = \text{“}\in U\text{”} \,/ large = \text{“}\in U\text{”}$

It is difficult to trace back to the first appearance of the concept of a universe. Essentially, it captures the idea of a collection closed under certain operations. But why introduce universes in the context of set-theoretic foundations of category theory? As Shulman (2008) suggests, we can reason as follows: on an *informal* level what we need for freely manipulating large categories seems to be a theory of classes which resembles closely ZFC; in practice it should be enough to have two copies of the axioms of ZFC, once for sets, once for classes. On a *formal* level, **universes** are introduced as a more elegant (and economic) solution to the same problem: instead of rewriting twice the axioms of ZFC we identify specific sets in our system as good candidates to interpret *large* collections.

5.1 Grothendieck's universes

As the name of this subsection suggests, the use of *universes* as foundational recipe for category theory goes back to Grothendieck. The purpose of his project, closely related to Bourbaki, was to justify the use of category theory in mathematical practice (and in Grothendick's perspective specifically in Algebraic Geometry).

Here is the definition of universe:[25]

[23] See here definition 2.

[24] $Def(X)$ denotes the set of all the subsets definable from element of X, i.e. sets of the form $\{x \in X | \varphi(x)\}$, where $\phi(x)$ can contain parameters from X and all its quantifiers range only over elements of X.

[25] In the original presentation (Bourbaki (1972), p. 185) the definition of universe also includes closure under ordered pairs which are a primitive notion in Bourbaki's presentation.

Definition 5. *A set U is a **Grothendieck universe** if the following conditions hold:*

(i) $if y \in x \in U, then y \in U;$

(ii) $if x, y \in U, then \{x, y\} \in U;$

(iii) $if x \in U, then \wp(x) \in U;$

(iv) *if* $(x_i)_{i \in I}$ *is a family of elements of* U*, and* $I \in U$*, then* $\bigcup_{i \in I} x_i \in U$.

In words, the definition says that U is a Grothendick universe if it is a transitive set closed under pairs,[26] power set and union of elements of U indexed by an element of U. In a more set-theoretical flavour, we can describe this definition as requiring $U = V_\kappa$ for some inaccessible cardinal κ (under the added hypothesis that U is uncountable[27]). Since inaccessible cardinals cannot be proved to exist in ZFC,[28] asserting the existence of a Grothendieck universe is a genuine strengthening of ZFC's axioms.

For a fixed Grothendieck universe U, we can rephrase our dichotomy between small and large by calling a category *large* whenever its collection of objects is a set *not belonging to* U. In case the universe is uncountable this is equivalent to assert that a category is small if and only if its collection of objects has *rank* less than κ, where κ is inaccessible.

This third approach, does not just give a satisfactory solution to conditions (A) and (B) (page 4), but it also allows for the construction of the category of functors between arbitrary categories (requirement C). In addition, it gives a more expressive semantics for the term *large*: we do not collapse every large collection to the size of V, as it happens in NBG, but we can retain a more careful distinction between *small*, *large* and *even larger* categories.

The following example gives an idea of the expressive power that we reach when introducing universes in the metatheory.

Example 7. *Every large category* $\mathbb{C}$ *has a category of presheaves* $Set^{\mathbb{C}^{op}}$*, and, if* $\mathbb{C}$ *is locally small*[29] *we can consider the Yoneda embedding* $y : \mathbb{C} \hookrightarrow Set^{\mathbb{C}^{op}}$.

Nevertheless we might want to be able to *encode* more abstract nonsense, and not satisfied by a single universe, we would like to have at our disposal a bigger universe U' (i.e. another inaccessible cardinal $\lambda > \kappa$), and then one other

[26]We do not assume as primitive the notion of ordered pair but, as usual, we define them à la Kuratowski: $(x, y) = \{\{x\}, \{x, y\}\}$.

[27]If we do not make any condition on the cardinality of U, also the emptyset, $\emptyset$, and ω are Grothendieck universes.

[28]A simple argument is the following: since V_κ, for κ inaccessible, represents a model of ZFC, if it was possible to prove the existence of such a cardinal in ZFC, then ZFC would also prove its own consistency, contradicting Gödel's second incompleteness theorem.

[29]See table on page 14 for the definition of a locally small category in presence of a universe.

above.[30] For this reason Grothendieck's initial proposal consisted of adding not just a single universe but an abundance. Formally we can express this by adding to the usual axioms of ZFC the following:

Grothendieck's axiom. *Every set is contained in some universe.*

This axiom guarantees the existence of sufficient large sets where every possible category we can meet is included.[31] Clearly, we have moved far from the strength of ZFC: the system obtained by adding Grothendieck's axiom to ZFC has the same consistency as ZFC + "there exist inaccessible cardinals of arbitrary size". As noticed by Mac Lane[32] this axiom does not solve definitely all the problems. We do not have any a priori certainty that changing universe does not affect the construction of our categories, or preserves all the properties of a specific object. Consider the following example

Example 8. *Let us assume that we have proved, for some property ϕ, the existence of a group G such that $\phi(G, H)$ is true for every small group H (for example ϕ could tell us that G is the limit of some diagram in* Grp*). The same argument still holds if we interpret the notion of largeness with some specific inaccessible κ, but there is no guarantee that the group G will be the same under all the possible interpretations.*

As kindly pointed out by one of the anonymous referees, in order to obtain this stronger property we should ask for the universe U to be an elementary substructure of V. For this, stronger axioms of infinity are needed, namely we have to ask for the cardinality of the universe to be at least a Mahlo number. The introduction of such large cardinals can be related to a general reflection principle for ZFC.[33] Even if the existence of these cardinals are given by axioms stronger than the one asserting the existence of a single inaccessible, and also stronger than Grothendieck's axiom, these axioms are still quite "weak" if compared to current large cardinal axioms used by set theorists. A similar approach based on a general reflection principle has been sketched by Engeler and Röhrl (1969). The following quotation concludes the paragraph where the two authors describe their proposal:[34]

> [...] However, the main objection to this approach is quite indepen-

[30] One possible reason is that we do not want just to consider the category of all small categories but also the category of all large ones, or of all locally small ones...

[31] As Shulman (2008) notes, this axiom asserts the possibility to enlarge the universe, more than asserting the existence of multiple stratified universes.

[32] See Lane 1969, p. 2.

[33] The interested reader should consult Lévy (1960). We will come back on a much weaker formulation of the reflection principle in section 5.3.

[34] See E. Engler (1969), p. 62.

dent of the strength and questionability of the additional assumptions creating universes. We believe that it is a faulty to make a procrustes bed of set theory and try, bend or break, to fit all mathematical structures into it. This does injustice, in particular to category theory, as it denies the autonomous role that such theories play in mathematics.

To conclude our survey of the use of universes as foundation for category theory, we can sum up the situation with the following table:[35]

small	$Morph(\mathbb{C}) \in U$
locally small	$\forall c, d \in Obj(\mathbb{C}) \in U$ $Hom_{\mathbb{C}}(c, d) \in U$
large	$Morph(\mathbb{C}) \subseteq U$, $Morph(\mathbb{C}) \notin U$
enormous	$Morph(\mathbb{C}) \nsubseteq U$

5.2 Mac Lane's proposal

> [...] It turns out that a flexible and effective formulation of the present notions of category theory can be given with a more modest addition to the standard axiomatic set theory: the assumption that there is **one** universe.
>
> Saunders Lane (1969), p. 193.

As we have already mentioned in the last paragraph, one of the first mathematician who highlighted some problems of the foundational approach proposed by the French school of Grothendieck was Saunders Mac Lane, one of the founders of category theory.

In 1969 Mac Lane published a paper with a meaningful title: *One universe for the foundations of category theory*. In this work he argues that the existence of a single universe in ZFC is sufficient to have a foundational framework for

[35]Observe that we can always identify objects of $\mathbb{C}$ with identity morphisms. In this table we indicate with *Morph*($\mathbb{C}$) the collection of morphisms of a category $\mathbb{C}$, and with $Hom_{\mathbb{C}}(c,d)$ the set of morphisms between two given objects c, d of $\mathbb{C}$.

category theory. His proposal essentially consists in weakening Grothendieck's axiom asking, not for an abundance of universes, but just one.

Mac Lane defines a universe as follows:

Definition 6. *A set U is called a universe if:*

(1) $x \in y \in U implies x \in U$;

(2) $\omega \in U$;

(3) $x \in U implies \wp(x) \in U$;

(4) $x \in U implies \bigcup x \in U$;

(5) if $f : x \to y$ *is a surjective function such that* $x \in U$ *and* $y \subset U$*, then* $y \in U$.

As Mac Lane notices, the conjunction of condition (4) and (5) is equivalent to condition (*iv*) in definition 5. Apart from this and the requirement that U is uncountable (condition (2) and (3)), the definition is the same as that given by Bourbaki.[36]

In this framework the systematization of the dichotomy *small/large* is essentially the same as that of Grothendieck's school (See table on page 14). The restriction to a single universe allows for a (almost[37]) complete treatment of category theory and, at the same time, allows us to escape from the "jungle" of multiple universes.[38]

Finally we remark that consistency of Mac Lane's proposal amounts to the consistency of ZFC + "there exists a strong inaccessible cardinal".

5.3 Feferman's proposal

Foundations of category theory have represented a problem of major interest for Solomon Feferman, who came back to this topic several times during the last forty years. He dedicated four papers[39] to this issue, proposing more than a single solution. Here we confine ourselves to the analysis of his first proposal.

The first paper where Feferman addresses the question was published in 1969.[40] In this work he proposes an alternative to the solution of adopting new axioms for universes. Feferman's idea consists in using a well-known principle of set theory, namely the *reflection principle*.

[36] We also recall the treatment of ordered pairs as a primitive entity, characteristic of Bourbaki's approach.

[37] This approach does not allow for the construction of the category of all large categories.

[38] As remarked by one of the anonymous referee, the request of a single universe U inside V could be seen as a kind of opprobrium from the point of view of a set theoretician. An alternative solution to the universe juggling has been mentioned at the end of the last section: see for example E. Engler (1969).

[39] Namely Feferman (1969), Feferman (1977), Feferman (2006), Feferman (2004).

[40] See Feferman (1969).

Feferman's system, which we indicate as ZFC/s,[41] consists, in the first instance, in adding a new constant symbol s to the usual language of ZFC. Secondly we add to the axioms of ZFC further axioms in order to describe (the interpretation of) s as a natural model of ZFC.[42]

Before giving the axioms we recall that if φ is a formula of the language of ZFC, its relativization to s, denoted by φ^s , is given when all the quantifiers that occur in φ are bounded by s.[43]

Definition 7. *The system ZFC/s is given in the language $\mathcal{L}$ of ZFC extended with the constant symbol s by the following axioms:*

(1) Axioms of ZFC: extensionality, emptyset, pairs, union, powerset, infinity, foundation, replacement, choice.

(2) s is not empty:

$$\exists x(x \in s)$$

(3) s is transitive:

$$\forall x, y(y \in x \wedge x \in s \rightarrow y \in s)$$

(4) s is closed under subsets:

$$\forall x, y(x \in s \wedge \forall z(z \in y \rightarrow z \in x) \rightarrow y \in s)$$

(5) reflection axioms: for every formula φ with free variable $x_1,\ldots, x_n$:

$$\forall x_1 \ldots \forall x_n(\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi^s(x_1, \ldots, x_n))$$

The axiom schema (5) can be read in model-theoretic terms as follows: let (M, ϵ, S) be a model of ZFC/s,[44] call M_s the set $\{x \in M | x \epsilon S\}$, and ϵ_s the restriction of ϵ to M_s,[45] , then (M, ϵ) is an elementary extension of (M_s, ϵ_s). In other words the two models satisfy the same formula in the language $\mathcal{L}$.

As we mentioned, this axiom schema, is based on the reflection principle. A specific istance of this principle can be suitably reformulated as a theorem of ZFC. It might be helpful to highlight the common point this *reflection theorem* shares with the downward Löwenheim-Skolem theorem. The proof of the latter shows, given a model N of a theory T and an infinite subset $S \subset N$, how to

[41] the symbol s stands for *smallness*.

[42] A natural model of ZFC, (M, ϵ), is a transitive model of ZFC, closed under subsets: $x \subset y \in M$ implies $x \in M$.

[43] For example, the relativization to s of the formula $\forall a \exists b \forall x(x \in b \leftrightarrow \psi(x, a))$ is

$$\forall a \in s \exists b \in s \forall x \in s(x \in b \leftrightarrow \psi s(x, a)).$$

[44] We indicate with S the element of M which interprets the constant symbol s.

[45] i.e. $x \epsilon_s y$ iff $x, y \in M_s$ and $x \epsilon y$.

build a model M, such that $M \prec N$ (M is an elementary substructure of N) and $|N| = |S|$. In order to obtain the model M we build a sequence of sets M_n in this way: starting from $M_0 = S$ every M_{n+1} is obtained from M_n by adding a witness $b \in N$ for every existential sentence $\exists y\phi(y, x_1, \ldots, x_n)$ and every n-tuple of elements $a_1, \ldots, a_n \in M_n$ such that $\exists y \in N\phi N(y, a_1, \ldots, a_n)$ is a true sentence. M is then obtained as

$$M = \bigcup_{n\in\omega} M_n.$$

Since there are just a countable amount of sentences ϕ, the cardinality of the various M_n never increases. Finally, the countable union of countable sets is still countable from which it follows that $|M| = |S|$. This construction can be rearranged to be carried out on the cumulative hierarchy of V_α's. Even if this method enables us "to build models of ZFC", this does not violate Gödel's second incompleteness theorem. Indeed even if we can reflect every finite conjunction of sentences of ZFC, we are not able to reflect at once a single infinite conjunction of sentences expressing that V_κ is a natural model of ZFC for a specific κ.

One of the main advantages of this "logical" approach consists exactly in this: the "formal description" of s as a "natural model of ZFC" is not sufficient to prove in ZFC that (the interpretation of) s is a natural model of ZFC. This, in fact, allows Feferman to prove the following important result in Feferman (1977):

Theorem 9. *ZFC/s is a conservative extension of ZFC.*

This result guarantees that we have not really strengthen our starting set theory; in particular, in categorical terms this means that all that can be proved in ZFC/s about *small* objects, even using *large categories*, can already be proved in ZFC. Now it should be sufficiently clear that interpreting *small* as "element of S" and *large* as "set not necessarily in S", what we have is an appropriate foundational framework where it is possible to interpret definitions and theorems of category theory.

As in the other cases we can evaluate the expressive power of Feferman's system using the conditions on page 4. While we can check that ZFC/s easily meet (A), (B), (D), the problem with functor categories noticed with other systems is also complicated in this case: we do not only have a limitation of size for the domain of the functors, but we also have to confine ourselves to consider those functor categories whose objects are S-definable. This is a consequence of the relativization of the replacement axioms to s, which can be considered to express the inaccessibility of s under all functions definable in $\mathcal{L}$.[46] In other words,

[46] See Feferman (1969), p. 208.

if we read the replacement axioms as saying that the image of a set under a class-function is still a set, their relativization can be rephrased as stating that the image of a set under a function-class *which is definable from elements of S* is still a set. This restriction, even if apparently innocuous, can have annoying consequences: for example we should change the notion of completeness (definition 2) in ZFC/s, requiring limits for "all small functors" (i.e functors definable from *S*) rather then for all functors with small domain.

5.4 Some comments on Feferman's proposal

Feferman has been one of the first mathematical-logicians to get interested in foundations of category theory: his motivation has been primarily to fill the gap between the rapid development of category theory and its proper systematization inside the mathematical edifice.

Initially a careful attitude led him to investigate the foundations of this theory with different systems of set theory. Only later he turned his attention to a critical analysis of a categorical foundation of all mathematics.[47]

The system proposed by Feferman in Feferman 1969 and the conservativity result over ZFC are of particular interest for a foundational analysis of category theory. The relevance of Feferman's contribution is well expressed in the words of Blass[48]

> This approach developed in Feferman (1969), has two advantages. First, the assumptions guarantee that, if we prove a theorem about small sets by using large categories, then the same theorem holds for arbitrary sets; [...]. Second, the assumptions do not really go beyond ZFC; any assertion in the first-order language, not mentioning κ,[49] that can be proved using these assumption can also be proved without them.

The second feature of Feferman's system mentioned by Blass is the conservativity result of the previous paragraph (theorem 9). This theorem highlights the "conventional" use of inaccessible cardinals when discussing set-theoretic foundation of category theory. As Shulman notes:

> [...] Thus we obtain a precise version of our intuition that the use of inaccessibles in category theory is merely for convenience: since many categorical proofs stated using inaccessibles can be formalized

[47] The main argument of his criticism for a possible categorical foundation of all math was firstly formulated in his '77 paper Feferman (1977). He then came back to the same argument in his successive works.

[48] Blass (1984), page 8.

[49] Blass uses κ to indicate the level of the cumulative hierarchy which corresponds to the interpretation of the added constant symbol s in Feferman's system.

> in ZFC/s, any *consequence* of such a theorem not referring explicitly to inaccessibles is also provable purely in ZFC.

Even if inaccessible cardinals, and in general stronger axioms of infinity, have became part of modern mathematical research, their use in foundational contexts remains dubious. Again, in the words of the philosopher Marquis:[50]

> Any reference to inaccessibles is simply removed. This is an exact formulation of the conviction that questions of size are only used to justify certain general construction and they do not bear on the real mathematical content of the constructions and its consequences. [...] Feferman's results[51] are important for they can be interpreted as showing that *as far as set theory is concerned*, category theory does not raise *new* foundational problem.

[50] Marquis (2009), pp. 183-184.
[51] The reference is to Feferman (1969).

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