Gentzen Systems and Beth Tableaux: A Comparative Analysis of Syntactic and Semantic Proof Methods

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Abstract

This paper provides an expository comparison of two foundational proof systems in classical propositional logic: Gentzen's sequent calculus and Beth's semantic tableaux. Gentzen's sequent calculus is presented as a rule-based system built upon the single axiom $\alpha \Rightarrow \alpha$, whose key feature—the subformula property—guarantees that every provable formula admits a proof constructed entirely from its subformulas. Beth tableaux are introduced as a complementary, refutation-based method that establishes validity by decomposing a formula from its main connective into subformulas and deriving contradictions across all branches of a truth-value analysis. The correspondence between the two systems is demonstrated through parallel proofs of classical tautologies.

Keywords

sequent calculus, Beth tableaux, semantic tableaux, subformula property, proof theory

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Introduction

The early 20th century marked a period of profound transformation in logic and the foundations of mathematics, driven by a need for more rigorous, transparent proof systems. Two of the most significant figures in this movement were Gerhard Gentzen (1909–1945) and Evert Willem Beth (1908–1964).

Gentzen, a German mathematician, fundamentally reshaped modern proof theory through his development of natural deduction and the sequent calculus. The Cut-Elimination Theorem, also known as the Hauptsatz, established that every provable formula admits a proof constructed entirely from its subformulas. He further applied these methods to provide a consistency proof for Peano arithmetic.

Contemporaneously, Evert Willem Beth, a Dutch logician and philosopher, made critical strides in clarifying the relationship between syntax and semantics. His most influential contribution, the Beth Definability Theorem, established a deep connection between implicit and explicit definability in first-order logic. Furthermore, Beth developed and popularized semantic tableaux—a proof technique based on table-like semantic constructions. Unlike axiomatic and sequent-based systems, semantic tableaux work directly with the concepts of truth and falsity within models.

Gentzen's Sequent Calculus

Gentzen sought a calculus that maintained a "logistic" architecture—akin to his 1932 system, which eschewed assumption formulas in contrast to natural deduction—while simultaneously preserving the introduction and elimination bifurcation characteristic of the system NI. The sequent calculus realized this objective by formalizing logical consequence as a primitive structural rule via the operations of cut and weakening (thinning). Consequently, this structural isolation decoupled the operational rules, enabling them to remain purely analytic and strictly govern the behavior of individual connectives.

Language and Notation

For the purposes of this analysis, the language of the propositional calculus is defined as follows:

- $\alpha, \beta, \gamma, \delta$ denote formulas,
- Γ, Δ, Θ, Λ denote sets of formulas,
- $\supset$ denotes logical implication,
- $\Rightarrow$ denotes the Gentzen sequent arrow (derivability, $\vdash$),
- a sequent takes the form $\Gamma \Rightarrow \Delta$, $\Rightarrow \Delta$, $\Gamma \Rightarrow$, $\Rightarrow$. A formula is a Gentzen theorem if the antecedent of the sequent is empty.

Definition 1 (Subformula). *The subformulas of a formula α are those formulas that occur during its construction, including α itself.*

Rules of Gentzen's Calculus

The Gentzen system operates on a single identity axiom: $\alpha \Rightarrow \alpha$. The deductive rules governing the calculus are categorized as

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follows:

- weakening rule $\dfrac{\Gamma \Rightarrow \Theta}{\alpha, \Gamma \Rightarrow \Theta} \quad \vee \quad \dfrac{\Gamma \Rightarrow \Theta}{\Gamma \Rightarrow \Theta, \alpha}$
- structural rules:
  - contraction $\frac{\alpha,\alpha,\Gamma\Rightarrow\Theta}{\alpha,\Gamma\Rightarrow\Theta} \quad \vee \quad \frac{\Gamma\Rightarrow\Theta,\alpha,\alpha}{\Gamma\Rightarrow\Theta,\alpha}$
  - exchange $\frac{\Lambda,\alpha,\beta,\Gamma\Rightarrow\Theta}{\Lambda,\beta,\alpha,\Gamma\Rightarrow\Theta} \quad \vee \quad \frac{\Lambda\Rightarrow\Gamma,\alpha,\beta,\Theta}{\Lambda\Rightarrow\Gamma,\beta,\alpha,\Theta}$
- logical rules:
  - conjunction introduction $\frac{\Gamma\Rightarrow\Theta,\alpha \qquad \Gamma\Rightarrow\Theta,\beta}{\Gamma\Rightarrow\Theta,\alpha\wedge\beta}$
  - conjunction elimination $\frac{\alpha,\Gamma\Rightarrow\Theta}{\alpha\wedge\beta,\Gamma\Rightarrow\Theta} \quad \vee \quad \frac{\beta,\Gamma\Rightarrow\Theta}{\alpha\wedge\beta,\Gamma\Rightarrow\Theta}$
  - disjunction introduction $\frac{\alpha,\Gamma\Rightarrow\Theta \qquad \beta,\Gamma\Rightarrow\Theta}{\alpha\vee\beta,\Gamma\Rightarrow\Theta}$
  - disjunction elimination $\frac{\Gamma\Rightarrow\Theta,\alpha}{\Gamma\Rightarrow\Theta,\alpha\vee\beta} \quad \vee \quad \frac{\Gamma\Rightarrow\Theta,\beta}{\Gamma\Rightarrow\Theta,\alpha\vee\beta}$
  - negation introduction $\frac{\alpha,\Gamma\Rightarrow\Theta}{\Gamma\Rightarrow\Theta,\neg\alpha} \quad \vee \quad \frac{\Gamma\Rightarrow\Theta,\alpha}{\neg\alpha,\Gamma\Rightarrow\Theta}$
  - implication $\frac{\alpha,\Gamma\Rightarrow\Theta,\beta}{\Gamma\Rightarrow\Theta,\alpha\supset\beta} \quad \vee \quad \frac{\Gamma\Rightarrow\Theta,\alpha \qquad \beta,\Lambda\Rightarrow\Delta}{\alpha\supset\beta,\Gamma,\Lambda\Rightarrow\Theta,\Delta}$

The central metamathematical result concerning Gentzen's sequent calculus is the following:

Theorem 1 (Subformula Property). *If a sequent $\Gamma \Rightarrow \Delta$ has a proof in the sequent calculus, then it has a proof in which every formula occurring in the derivation is a subformula of some formula in $\Gamma \cup \Delta$.*

This theorem is a direct consequence of Gentzen's cut-elimination theorem (Hauptsatz). Its significance lies in the fact that it transforms the search for proofs into a finite, bounded process: one need only consider combinations of subformulas of the target sequent.

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Corollary 1. *The propositional sequent calculus is decidable. To determine whether a sequent $\Gamma \Rightarrow \Delta$ is provable, one systematically examines all possible derivations constructed from subformulas of formulas in $\Gamma \cup \Delta$. If no such derivation terminates in an instance of the identity axiom $\alpha \Rightarrow \alpha$, then the sequent is not provable.*

The Structural Characterization of Classical and Intuitionistic Logics in Gentzen's Calculus

(i) The Architectural Asymmetry of Classical Natural Deduction (NK)

In the framework of Natural Deduction, intuitionistic logic (NJ) exhibits a high degree of proof-theoretic harmony, wherein the introduction and elimination rules for each logical connective are perfectly balanced and dual to one another. Conversely, classical natural deduction (NK) is inherently asymmetric. Accommodating classical logic requires the ad hoc appendage of a non-constructive axiomatic rule, such as Double Negation Elimination:

$$\neg\neg A \to A$$

Because this rule violates the pure introduction/elimination paradigm, it disrupts the subformula property and breaks Gentzen's normalization procedure, rendering the system proof-theoretically unruly.

(ii) The Sequent Calculus as a Symmetric Solution (LK vs. LI)

To restore structural symmetry to classical logic, Gentzen engineered the sequent calculus (LK). He demonstrated that classical logic could achieve elegant syntactic symmetry by expanding the geometric structure of the sequent itself—specifically, by permitting multiple formulas to populate the succedent (the right-hand side of the turnstile):

$$\Gamma \vdash \Delta$$

By introducing multi-conclusion sequents, the operational rules for negation ($\neg$) achieve a perfect, symmetric duality. This structural innovation ultimately enabled Gentzen to prove his foundational Hauptsatz (the Cut-Elimination Theorem) uniformly for classical logic.

(iii) Intuitionistic Logic as a Structural Restriction (*LI*)

Crucially, Gentzen did not need to construct an entirely disparate calculus to formalize intuitionistic logic (*LI*). Instead, he demonstrated that intuitionistic logic emerges naturally from the classical sequent calculus (*LK*) via a strict structural constraint: the succedent must be restricted to at most one formula:

$$\Gamma \vdash A \quad \text{or} \quad \Gamma \vdash \emptyset$$

Consequently, Gentzen established that the divergence between classical and intuitionistic systems does not reside in the semantic or operational definitions of the logical connectives themselves, but rather in the underlying structural permissions—specifically, the capacity to support multiple concurrent conclusions.

Beth Tableaux

Beth tableaux constitute a systematic, tabular method for establishing the validity of logical formulas. The method operates by

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indirect proof: to show that a formula ϕ is valid, one assumes that ϕ is false and aims to demonstrate a contradiction in every possible case.

The method begins by assuming the negation of the formula. The formula is then decomposed according to its main logical connective, distributing its components into the "True" and "False" columns in accordance with the semantic conditions for the connective. When the decomposition of a formula requires a case analysis — as occurs, for instance, with implication on the "True" side — the tableaux branches into two sub-tableaux, each of which must be independently closed.

The Semantic Divide (Bivalence vs. Constructive Information)

(i) Classical Semantic Tableaux and the Principle of Bivalence

In the classical paradigm, semantic tableaux are grounded in the principle of bivalence, which dictates that every proposition possesses exactly one of two static truth values—Truth or Falsity—relative to a completed, immutable state of affairs. Consequently, a classical tableaux functions as a systematic, algorithmic search for a classical countermodel. Because classical semantics are entirely truth-functional and static, the branches of a classical tableaux delineate truth-value assignments across a single model. From a proof-theoretic standpoint, this framework serves as the exact semantic dual to Gentzen's classical sequent calculus (*LK*).

(ii) Intuitionistic Tableaux and Constructive States of Knowledge

Conversely, intuitionistic logic rejects the principle of biva-

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lence, replacing the notion of mind-independent truth with the concepts of assertibility, verification, or constructive proof. As a result, an intuitionistic countermodel cannot be framed as a static assignment of truth values.

To resolve this, Evert Willem Beth recognized that constructing a tableaux system for intuitionistic logic required formalizing an asymmetric, dynamic flow of information. Within an intuitionistic semantic tableaux, a branch does not signify an immediate assignment of "True" or "False"; rather, it tracks evolving states of knowledge that expand monotonically over time.

The Relationship Between Beth Tableaux and Gentzen's Sequent Calculus

The two proof methods can be understood as inverse procedures operating on the same underlying logical structure.

Definition 2 (Signed Formula). *A* signed formula *is an expression $T\alpha$ or $F\alpha$, where α is a formula. Intuitively $T\alpha$ asserts "α is true" and $F\alpha$ asserts "α is false."*

Definition 3 (Branch and Sequent Translation). *A tableaux branch B is a set of signed formulas. Define*

$$\begin{aligned} \mathrm{Seq}(B) \; &= \; \Gamma_B \Rightarrow \Delta_B, \\ \Gamma_B &= \{\alpha : T\alpha \in B\}, \\ \Delta_B &= \{\alpha : F\alpha \in B\}. \end{aligned}$$

So the "True" column of a branch is read as the antecedent of a sequent, and the "False" column as the succedent.

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The tableaux for a formula φ is initialised with the single branch $B_0 = \{F\varphi\}$, so $\mathrm{Seq}(B_0) = \Rightarrow \varphi$ — exactly the sequent Gentzen's calculus is trying to derive. This already signals the inversion: the tableaux *starts* at the target sequent $\Rightarrow \varphi$ and works outward by decomposition, while Gentzen's calculus *starts* at atomic axioms and works inward by composition, arriving at $\Rightarrow \varphi$ only at the very end.

Definition 4 (Closure). *A branch B is* closed *iff there is an atomic p with $Tp, Fp \in B$. A tableaux is closed iff every branch is closed.*

The Rule Correspondence Lemma

Each tableaux expansion rule decomposes a signed compound formula into signed subformulas, either extending the branch (non-branching, so-called α-type rules) or splitting it into two branches (branching, β-type rules). We show each such rule is exactly a Gentzen logical rule, read in the opposite direction.

Lemma 1 (Rule Correspondence). *Let B be a branch containing a signed compound formula, and let the tableaux rule for that formula extend B to B' (non-branching case) or split B into B_1, B_2 (branching case). Then:*

- ***α-type (non-branching) rules** — $T(\alpha \wedge \beta)$, $F(\alpha \vee \beta)$, $F(\alpha \supset \beta)$, $T\neg\alpha$, $F\neg\alpha$ — satisfy:* $\mathrm{Seq}(B)$ *is exactly the* conclusion *of a one-premise Gentzen logical rule, and* $\mathrm{Seq}(B')$ *is exactly its* premise.

- ***β-type (branching) rules** — $F(\alpha \wedge \beta)$, $T(\alpha \vee \beta)$, $T(\alpha \supset \beta)$ — satisfy:* $\mathrm{Seq}(B)$ *is exactly the conclusion of a two-premise*

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Gentzen logical rule, and $\mathrm{Seq}(B_1), \mathrm{Seq}(B_2)$ *are exactly its two premises.*

Proof. We check each connective; the correspondence is a direct unpacking of the two definitions.

Conjunction. $T(\alpha \wedge \beta)$ adds $T\alpha, T\beta$ to the same branch. So $\mathrm{Seq}(B) = \Gamma, \alpha \wedge \beta \Rightarrow \Theta$ while $\mathrm{Seq}(B') = \Gamma, \alpha, \beta \Rightarrow \Theta$ (the retained copy of $\alpha \wedge \beta$ is absorbed by contraction/weakening, which are admissible structural moves). This is exactly Gentzen's conjunction-elimination rule

$$\frac{\Gamma, \alpha, \beta \Rightarrow \Theta}{\Gamma, \alpha \wedge \beta \Rightarrow \Theta}$$

$F(\alpha \wedge \beta)$ splits into B_1 (adds $F\alpha$) and B_2 (adds $F\beta$). So $\mathrm{Seq}(B) = \Gamma \Rightarrow \Theta, \alpha \wedge \beta$, while $\mathrm{Seq}(B_1) = \Gamma \Rightarrow \Theta, \alpha$ and $\mathrm{Seq}(B_2) = \Gamma \Rightarrow \Theta, \beta$. This is exactly conjunction-introduction

$$\frac{\Gamma \Rightarrow \Theta, \alpha \qquad \Gamma \Rightarrow \Theta, \beta}{\Gamma \Rightarrow \Theta, \alpha \wedge \beta}$$

Disjunction is dual to conjunction by the De Morgan symmetry between the True/False columns, and the check is identical with the roles of the two columns exchanged: $T(\alpha \vee \beta)$ is β-type and matches disjunction-elimination (two premises $\alpha, \Gamma \Rightarrow \Theta$ and $\beta, \Gamma \Rightarrow \Theta$); $F(\alpha \vee \beta)$ is α-type and matches disjunction-introduction.

Implication. $T(\alpha \supset \beta)$ splits into B_1 (adds $F\alpha$) and B_2 (adds $T\beta$), reflecting the truth condition of material implication. So $\mathrm{Seq}(B) = \alpha \supset \beta, \Gamma, \Lambda \Rightarrow \Theta, \Delta$, $\mathrm{Seq}(B_1) = \Gamma \Rightarrow \Theta, \alpha$, $\mathrm{Seq}(B_2) = \beta, \Lambda \Rightarrow \Delta$. This is exactly the implication-elimination rule

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$$\frac{\Gamma \Rightarrow \Theta, \alpha \qquad \beta, \Lambda \Rightarrow \Delta}{\alpha \supset \beta, \Gamma, \Lambda \Rightarrow \Theta, \Delta}$$

$F(\alpha \supset \beta)$ adds $T\alpha$ and $F\beta$ to the same branch (no branching, since falsifying $\alpha \supset \beta$ requires both α true *and* β false simultaneously). So $\mathrm{Seq}(B) = \Gamma \Rightarrow \Theta, \alpha \supset \beta$ and $\mathrm{Seq}(B') = \alpha, \Gamma \Rightarrow \Theta, \beta$. This is exactly implication-introduction

$$\frac{\alpha, \Gamma \Rightarrow \Theta, \beta}{\Gamma \Rightarrow \Theta, \alpha \supset \beta}$$

Negation. $T\neg\alpha$ (i.e. $\neg\alpha \in \Gamma$) replaces it with $F\alpha$ in the same branch: $\mathrm{Seq}(B) = \neg\alpha, \Gamma \Rightarrow \Theta$, $\mathrm{Seq}(B') = \Gamma \Rightarrow \Theta, \alpha$. This is exactly negation-elimination

$$\frac{\Gamma \Rightarrow \Theta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Theta}$$

$F\neg\alpha$ (i.e. $\neg\alpha \in \Delta$) replaces it with $T\alpha$: $\mathrm{Seq}(B) = \Gamma \Rightarrow \Theta, \neg\alpha$, $\mathrm{Seq}(B') = \alpha, \Gamma \Rightarrow \Theta$. This is exactly negation-introduction

$$\frac{\alpha, \Gamma \Rightarrow \Theta}{\Gamma \Rightarrow \Theta, \neg\alpha}$$

In every case the identification is exact — not merely analogous — because both definitions unpack to literally the same set-theoretic data. □

Note the branching/non-branching pattern lines up perfectly with the one-premise/two-premise pattern of Gentzen's rules: **a tableaux branches exactly where, and only where, the corresponding Gentzen rule has two premises.** This is the structural content of "the same underlying logical structure."

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Reduction to Atomic Axioms

Gentzen's calculus is stated with the axiom schema $\alpha \Rightarrow \alpha$ for arbitrary α, while a tableaux branch only ever closes at an *atomic* clash Tp, Fp. For the correspondence to be exact we must show this is not a mismatch: the general axiom is dispensable.

Lemma 2 (Atomic Identity). *For every formula α, the sequent $\alpha \Rightarrow \alpha$ is derivable in Gentzen's calculus using only* atomic *instances of the identity axiom (together with the logical and structural rules).*

Proof. Induction on the complexity of α.

Base case. α atomic: immediate, this is already an atomic instance.

Case $\alpha = \beta \wedge \gamma$. By hypothesis, $\beta \Rightarrow \beta$ and $\gamma \Rightarrow \gamma$ are derivable from atomic axioms. Weaken each: $\beta, \gamma \Rightarrow \beta$ and $\beta, \gamma \Rightarrow \gamma$. Combine by conjunction-introduction to get $\beta, \gamma \Rightarrow \beta \wedge \gamma$, then apply conjunction-elimination (adding both conjuncts at once) to obtain $\beta \wedge \gamma \Rightarrow \beta \wedge \gamma$.

Case $\alpha = \neg\beta$. By hypothesis $\beta \Rightarrow \beta$ is derivable from atomic axioms.

Step 1 — apply negation-introduction to $\beta \Rightarrow \beta$ (viewed as premise $\beta, \emptyset \Rightarrow \{\beta\}$) to get $\Rightarrow \beta, \neg\beta$.

Step 2 — apply negation-elimination to $\Rightarrow \beta, \neg\beta$ (viewed as premise $\emptyset \Rightarrow \{\neg\beta\}, \beta$) to get $\neg\beta \Rightarrow \neg\beta$.

Cases $\alpha = \beta \vee \gamma$ and $\alpha = \beta \supset \gamma$ are entirely symmetric to the $\wedge$ and $\neg$ cases, using the dual introduction/elimination pair for each connective. ◻

Consequently every Gentzen derivation can be normalised so that its leaves are atomic axiom instances (possibly weakened) — exactly the shape a tableaux closure produces. There is no loss of generality, and no gap, in comparing tableaux closure only to *atomic* axioms.

Main Theorem

Theorem 2 (Duality of Beth Tableaux and Gentzen's Sequent Calculus). *For every formula φ, the following are equivalent:*

(i) $\Rightarrow \varphi$ *is derivable in Gentzen's sequent calculus;*

(ii) the Beth tableaux rooted at $\{F\varphi\}$ *is closed.*

Proof. **(ii) $\Rightarrow$ (i).** Induct on the closed tableaux, from its leaves toward its root.

Base. A closed leaf-branch B has $\mathrm{Seq}(B) = \Gamma \Rightarrow \Theta$ with some atomic p in both Γ and Θ. By weakening, this sequent is derivable from the atomic axiom $p \Rightarrow p$.

Step. Suppose B is an internal branch. In the α-type case, its child B' already has a derivation of $\mathrm{Seq}(B')$; in the β-type case, its children B_1 and B_2 already have derivations of $\mathrm{Seq}(B_1)$ and $\mathrm{Seq}(B_2)$, by the induction hypothesis. By Lemma 1, $\mathrm{Seq}(B)$ is precisely the conclusion of the Gentzen rule whose premise(s) are $\mathrm{Seq}(B')$ (or $\mathrm{Seq}(B_1)$, $\mathrm{Seq}(B_2)$). Applying that rule to the derivation(s) already built extends them to a derivation of $\mathrm{Seq}(B)$.

Iterating up to the root $B_0 = \{F\varphi\}$, we obtain a derivation of $\mathrm{Seq}(B_0) = \Rightarrow \varphi$. Every step used is a genuine logical or structural rule of Gentzen's calculus, so this derivation is in fact cut-free.

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(i) ⇒ (ii). Suppose $\Rightarrow \varphi$ is derivable. By the Hauptsatz, it has a cut-free derivation D, all of whose formulas—by the subformula property—are subformulas of φ. By Lemma 2, D may further be normalised so its axiom leaves are atomic.

Induct on D from its root (endsequent $\Rightarrow \varphi$) down to its leaves. At each rule instance with conclusion Seq and premise(s) Seq′ (or Seq_1, Seq_2), Lemma 1 read in reverse identifies a tableaux branch B with $\mathrm{Seq}(B) = \mathrm{Seq}$ that expands, via the matching tableaux rule, to a branch B' with $\mathrm{Seq}(B') = \mathrm{Seq}'$ (or to branches B_1, B_2 with $\mathrm{Seq}(B_1) = \mathrm{Seq}_1$, $\mathrm{Seq}(B_2) = \mathrm{Seq}_2$). Building the tableaux in step with D's tree, the root of D (sequent $\Rightarrow \varphi$) yields the tableaux root $\{F\varphi\}$, and each axiom leaf $p \Rightarrow p$ of D (weakened to $\Gamma \Rightarrow \Theta$ with $p \in \Gamma \cap \Theta$) yields a branch containing both Tp and Fp — a closed branch.

Since D is a finite tree, the resulting tableaux is finite, and every branch terminates in closure. Hence the tableaux for φ closes.

Structure preservation. Both directions of the argument proceed by the *same* induction on the *same* finite tree, using Lemma 1 as the sole translation step between a tableaux node and a Gentzen rule instance, in each case matching branching to two-premise rules and non-branching to one-premise rules. Thus the closed tableaux for φ and a cut-free derivation of $\Rightarrow \varphi$ are literally the same tree: the tableaux is this tree grown downward from the target formula's negation by decomposition, and the Gentzen derivation is the identical tree grown upward from atomic axioms by composition. □

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Examples

To demonstrate the correspondence between Gentzen's sequent calculus and Beth tableaux, we present two classical propositional tautologies together with their proofs in both systems.

1) **Law of Transposition (Contraposition).**

$$(p \supset q) \supset (\neg q \supset \neg p)$$

Beth tableaux.

$$
\begin{array}{r|l}
\text{True} & \text{False} \\ \hline
 & (p \supset q) \supset (\neg q \supset \neg p) \\
(p \supset q) & (\neg q \supset \neg p) \\
\neg q & \neg p \\
p & q \\ \hline
\begin{array}{r|l} \text{True} & \text{False} \\ \hline & p \\ p & \end{array} &
\begin{array}{r|l} \text{True} & \text{False} \\ \hline q & \\ & q \end{array}
\end{array}
$$

Gentzen's proof.

$$
\begin{array}{rl}
p \Rightarrow p \qquad q \Rightarrow q & \\
p \supset q, p \Rightarrow q & (\supset) \\
p \supset q \Rightarrow q, \neg p & (\text{NI}\neg) \\
p \supset q \Rightarrow \neg p, q & (\Rightarrow \ \text{exchange}) \\
\neg q, p \supset q \Rightarrow \neg p & (\neg\text{NI}) \\
p \supset q \Rightarrow \neg q \supset \neg p & (\Rightarrow\supset) \\
\Rightarrow (p \supset q) \supset (\neg q \supset \neg p) & (\Rightarrow\supset)
\end{array}
$$

2) **The Law of Conditional Hypothetical Syllogism.**

$$(p \supset q) \supset [(q \supset r) \supset (p \supset r)]$$

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Beth tableaux.

<table>
<tr><th>True</th><th>False</th></tr>
<tr><td></td><td>$(p \supset q) \supset [(q \supset r) \supset (p \supset r)]$</td></tr>
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Gentzen's proof.

$$
\begin{array}{rl}
p \Rightarrow p \qquad q \Rightarrow q \qquad r \Rightarrow r & \\
p \Rightarrow p \qquad q, q \supset r \Rightarrow r & (\supset) \\
p \supset q, q \supset r, p \Rightarrow r & (\supset) \\
p \supset q, q \supset r \Rightarrow p \supset r & (\Rightarrow\supset) \\
p \supset q \Rightarrow (q \supset r) \supset (p \supset r) & (\Rightarrow\supset) \\
\Rightarrow (p \supset q) \supset [(q \supset r) \supset (p \supset r)] & (\Rightarrow\supset)
\end{array}
$$

Summary

The two proof methods can be understood as inverse procedures operating on the same underlying logical structure. Gentzen's sequent calculus constructs proofs in a bottom-up manner (when read from conclusion to premises): beginning with identity axioms of the form $\alpha \Rightarrow \alpha$, successive applications of logical and structural rules build increasingly complex sequents until the target sequent $\Rightarrow \phi$ is reached. In contrast, Beth tableaux operate in a top-down model: beginning with the target formula ϕ assumed false, the method decomposes ϕ into its constituent subformulas until atomic contradictions of the form $p \wedge \neg p$ are reached.

In both cases, the proof involves only subformulas of the target formula ϕ. The identity axiom $\alpha \Rightarrow \alpha$ of Gentzen's calculus corresponds directly to the closure condition of a tableaux branch, where a propositional variable appears on both the True and False sides.

The soundness of the correspondence between the two systems is ultimately grounded in the Deduction Theorem, which holds for both classical and intuitionistic logic:

Theorem 3 (Deduction Theorem). *For any set of formulas* Γ *and formulas* α, β*:*

$$\Gamma, \alpha \models \beta \text{ \textit{iff} } \Gamma \models \alpha \supset \beta$$

Consequently, the rules for constructing tableaux and the Gentzen inference rules both correspond to theorems of classical logic, ensuring that a formula is provable in one system if and only if it is provable in the other.

Conclusion

Both Gentzen's sequent calculus and Beth's semantic tableaux represent monumental advancements in proof theory, successfully resolving the opaque nature of earlier axiomatic systems. While Gentzen approached the problem syntactically—ensuring proofs could be built from subformulas without relying on external premises—Beth provided a highly visual, semantic method rooted in models of truth and falsity. As demonstrated, these systems are not merely parallel; they are complementary and inverse. The analytical decomposition of a formula via Beth tableaux provides a clear blueprint for the synthetic construction of a rigorous

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Gentzen proof, underscoring the deep unity between syntax and semantics in classical logic.

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